The 2-period Binomial Model

Introduction

Once we have understood the one period binomial model it is very easy to extend the model to two or more periods so that derivatives with maturity in two or more periods can be examined. We shall just examine the two-period model.

The Two-Period Model

The binomial model is extended by adding to new branches of the tree after each node. Proceeding in the same way as with the one period model after each node the price of the underlying asset either increases by a factor of u or decreases by a factor d.[1] Thus whether the value of the underlying after two periods is either $(1+u)^2S$ if the stock has gone up in two successive periods, $(1+d)^2S$ if the stock has gone down in two successive periods, $(1+u)(1+d)S$ if the stock first went up and then went down, or $(1+d)(1+u)S$ if the stock went down and then went up in the second period. Since $(1+u)(1+d) = (1+d)(1+u)$, the value of the stock is the same whether it first went up and then down or down and then up. Thus the two branches of the tree recombine after two periods and there are only three possible values for the value of the underlying after two periods. (After n periods there will be $n-1$ possible ending values for the underlying asset in such a *recombinant* tree).

The objective is to find the value of the option or derivative at the initial node of the tree. Let's consider an example with $u = 0.75$, $d = -0.25$, $S = 100$, $X = 100$ and $r = 0.25$ and extend it to two periods. The ending

[1] Remember that what is important is that $d < r < u$ and that it may be possible that $d > 1$.

values for the underlying asset are 306.25, 131.25 and 56.25. To value the call option at the initial node we first value the call at the final nodes and then work backward. The value of the call at the final nodes is simply 206.25, 31.25 and 0, that is the stock value minus the strike price. The value at the intermediate note then can than be computed using the methods of delta hedging, synthetic options or risk neutral valuation used in the section on the one period model. The method of risk neutral valuation is simple because the risk neutral probability is the same for each period as it depends only on u, d and r that are unchanging. Taking this risk-neutral valuation method, the value following the first up move is

$$\frac{\frac{1}{2}(206.25 + 31.25)}{(1 + \frac{1}{4})} = 95$$

and the value of the call option following a down movement in the stock is

$$\frac{\frac{1}{2}(31.25)}{(1 + \frac{1}{4})} = \frac{25}{2}.$$

Having found the value of the option after the first period, the same method can be used to find the value at the initial node:

$$\frac{\frac{1}{2}(95 + \frac{25}{2})}{(1 + \frac{1}{4})} = 43.$$

Risk Neutral Valuation and State Prices

The risk neutral valuation method also gives a very simple method of calculating the value of the option at the initial node. The risk neutral probability of two up movements in the underlying stock is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The probability of stock ending with a value of 131.25 is the probability of an up movement followed by a down movement *plus* the probability of a down movement followed by an up movement. Thus the risk neutral probability for this event is $\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$. Thus the value of the call option can also be evaluated directly as

$$\frac{\frac{1}{4}(206.25) + \frac{1}{2}(31.25)}{(1 + \frac{1}{4})^2} = 43.$$

Likewise the state prices are easily calculated (by dividing the risk neutral probabilities by $(1 + r)^2 = \frac{25}{16}$) to be $q_{uu} = \frac{4}{25}$, $q_{ud} = q_{du} = \frac{8}{25}$ and $q_{dd} = \frac{4}{25}$. Thus the call value could also easily be calculated as

$$\frac{4}{25}(206.25) + \frac{8}{25}(31.25) = 43.$$

American Options

As we have seen a European call option will have a positive time value as the lower bound for the European call is $S_t - X/(1+r) \geq S_t - X$ with inequality if $r > 0$. Since American options can't be worth less than equivalent European options it follows that American call options on non-dividend paying stock will not be exercised early because selling the option will always dominate exercising it. Intuitively an early exercise will involving paying the strike price earlier and this is undesirable. However, American put options can be exercised early because this will involve receiving the strike price earlier. The two period binomial model can be used to illustrate this possibility.

Consider a put option in our example with a strike price $X = 100$. The value of this put option at the final nodes is 0, 0 and 43.75. Thus the value of the put option following an up movement in the first period is 0 as the option can never get back in the money. However, following a down movement in the first period the value of the option is using the state prices $\frac{2}{5}(41.75) = 17.5$. But early exercise of the option would give 25. Thus early exercise is the better alternative and the option must have a value of 25 if it is of the American type. At the initial node the option is thus worth 10 if it is an American option and 7 if it is a European option that cannot be exercised early at the end of the first period.