One Period Binomial Model

These notes consider the one period binomial model to exactly price an option. We will consider three different methods of pricing an option: *delta-hedging*, creating a *synthetic option* using the underlying asset and the risk-free asset and calculating the *risk-neutral probabilities*. All methods will be used again when we extend the binomial model beyond one period and when we consider continuous trading.

Binomial Model

To value options exactly it is necessary to make specific assumptions about the determinants of the price of the underlying security. In practice this often means that a continuous stochastic process for the underlying is estimated or inferred from observations on past prices. However, the essential aspects can all be embodied in a simple binomial model in which the underlying asset can be one of two possible values at the end of each period; to put in starkly the asset can either go up or go down in value. By considering more than one period a *binomial tree* is constructed, where the price that an asset has branches out, either upward or downward at each point in time. These binomial trees can provide a remarkably good approximation to a more complex stochastic process provided that the number of periods is reasonably large, say above thirty.

In fact nearly all the basic principles of derivative pricing can be explained with a one or two period binomial model and this section develops the one period binomial model to study how options are exactly priced and the riskiness and elasticity of options.

Delta Hedging

Buying a stock is a risky investment. Buying a call option on that stock is even riskier. Yet combining the stock with the option can produce an investment that is risk-free. That is it is possible to hedge a position in the stock by taking an opposite position in the call option. To create such a risk-free investment it is necessary to buy the stock and the option in exactly the right proportions. The number of units of the stock required per option is known as the Δ (Delta) of the stock and taking these positions to create a risk-free investment is known as *Delta Hedging*.

We will now illustrate how to calculate Δ in a simple binomial example. Once Δ is known the option price itself can be calculated as any risk-free investment must in the absence of arbitrage opportunities earn the risk-free rate of return.

Consider a simple example where the price of the underlying stock is 100 and at the end of three months it has either risen to 175 or has fallen to 75. There is an "up" state and a "down" state. Let $u = 0.75$ and $d = -0.25$, so that the value of the stock in three months in the up state is $100(1+u) = 175$ and the value in the down state is $100(1 + d) = 75$. The net growth rate of the value of the stock in the up state is 75%, and the net growth in the value of the stock in the down state is -25%.

Since the stock can either rise by 75% or fall by 25% buying the stock is a risky investment. Now consider an at the money call option on this stock with a strike price of 100. If the stock goes up then in three months the call can be exercised for a profit of $175 - 100 = 75$ and if the price of the stock goes down, then the call will not be exercised. Thus the payoff to the call option is 75 in the up state and 0 in the down state. So the call is also risky. Suppose however, that we consider buying Δ units of the stock and writing one call. The payoff from this portfolio is $175\Delta - 75$ in the up state: the stock has gone up to 175 but the call option we have written will be

Position	Value Now	Value in down state	Value in up state
4 Short Calls	$-4c$	0	300
3 Long Shares	300	225	525
Overall	$300 - 4c$	225	225

Table 1: Risk-Free Portfolio - Example

exercised against us so we have an obligation of 75 The payoff in the down state is simply 75Δ as the call option is not exercised in the down state. To make this portfolio risk-free it is necessary to choose Δ such that

$$175\Delta - 75 = 75\Delta$$

so that the payoff is the same no matter whether we are in the up state or the down state. Solving for Δ gives $\Delta = 3/4$. Thus to completely hedge out the risk in the stock, we should sell four call options for every three units of stock we buy. The payoff from such a portfolio is given in Table 1. The value in three months time is 225 in both states and hence the portfolio is risk-free. Since it is risk-free it can be valued using the risk-free rate of interest. Suppose the risk-free rate of interest is 1/4 or 25%. The present value of this portfolio is $225/(1 + 1/4) = 180$. The value of the portfolio now is simply the current value of the share, $3 \times 100 = 300$ less $4c$ where c is the price of the call option. If there are no arbitrage possibilities then the value of this portfolio must also have a value of 3180:

$$300 - 4c = 180.$$

So solving for the call price gives $c = 30$. This shows that the Δ-hedge can be used to price the option.

The same procedure can be generalized. Let S be the value of the underlying stock, so its terminal value is $S_u = (1 + u)S$ in the up state and $S_d = (1 + d)S$ in the down state. Let K be the strike price of the option and $c_u = \max[0, S_u - K]$ be the value of the call option in the up state and

$c_d = \max[0, S_d - K]$ is the value of the call option in the down state. Remember, that the call option gives us the right to buy the underlying at a price of K, so for example in the up state when the stock is worth S_u, the option gives the right to buy at K and asset that can be sold for S_u. Thus the option will be exercised for a profit of $S_u - K$ if $S_u - K > 0$ and won't be exercised otherwise.

Now consider the Δ-hedge portfolio that writes one call and buys Δ units of the underlying. The payoffs from this portfolio are given in Table 2. The value of Δ is chosen so that the portfolio is riskless, $\Delta S_d - c_d = \Delta S_u - c_u$, i.e.

$$\Delta = \frac{c_u - c_d}{S_u - S_d} = \frac{c_u - c_d}{(u-d)S}.$$

The value of the risk-free portfolio is evaluated at the risk-free rate of interest r, so that

$$\Delta S - c = \frac{\Delta S_d - c_d}{(1+r)}.$$

Using the value of Δ just derived, and substituting $(1+d)S$ for S_d and solving for c gives after some manipulation

$$c = \frac{pc_u + (1-p)c_d}{(1+r)}$$

where $p = (r-d)/(u-d)$. Thus for any values of S, K, r, u and d, the value of the call option can be calculated from this formula.

There is a natural interpretation for Δ. It is the difference in the value of the call at maturity $c_u - c_d$ in the up state rather than the down state relative to the difference in the value of the stock $S_u - S_d$. It tells us the change in the value of the call relative to the change in the value of the stock or how much the value of the call changes when the stock changes by a given amount. Thus Δ tells us how responsive is the call value to changes in the value of the stock. We will return to this interpretation when we consider the elasticity of an option later on.

Position	**Value Now**	**Value in down state**	**Value in up state**
Short Call	$-c$	$-c_d$	$-c_u$
Δ Long Shares	ΔS	ΔS_d	$\Delta u S_u$
Overall	$\Delta S - c$	$\Delta S_d - c_d$	$\Delta S_u - c_u$

Table 2: Risk-Free Portfolio

Synthetic Options

The idea underlying Δ-Hedging is to create a risk-free asset using a combination of the stock and the option that can be valued at the risk-free rate of interest. The risk-free asset and the underlying asset can themselves be combined to replicate the payoffs of the option itself. Such a combination is called a *synthetic* option.

Consider again our simple example where the price of the underlying stock is 100 and at the end of three months it has either risen to 175 or has fallen to 75. Now consider how an at the money call option on this stock with a strike price of 100 can be *synthesized*. The payoff to the call option in the up state is 75 and the payoff in the down state is 0 as the option will not be exercised. Consider buying Δ units of the stock and investing B units of funds at the risk-free interest rate of 1/4. The payoff from this portfolio in the up state is $175\Delta + (1 + 1/4)B$ and the payoff in the down state is $75\Delta + (1 + 1/4)B$. To synthesize the option we must match these payoffs to the payoffs from the option. That is we must find Δ and B to solve

$$175\Delta + (1 + 1/4)B = 75$$

$$75\Delta + (1 + 1/4)B = 0.$$

Solving these two equations simultaneously gives $\Delta = 3/4$ and $B = -45$. Thus the option can be replicated by *borrowing* 45 and buying $\frac{3}{4}$ units of the stock. Since the stock costs 100 the the cost of buying 3/4 of a unit of stock is 75. Thus the net cost of synthesizing the option is the price we pay minus

Position	**Value Now**	**Value in down state**	**Value in up state**
Borrow B	-45	$225/4$	$225/4$
Buy Δ shares	75	$225/4$	$525/4$
Overall	30	0	75

Table 3: Synthetic Option

the amount borrowed, $75 - 45 = 30$. Again the portfolio that replicates or synthesizes the call must have the same price as the call itself. Hence $c = 30$ and we reach exactly the same conclusion as before. The payoffs are summarized in Table 3.

Again it is simple to generalize this procedure. To find the number of shares to be bought Δ and the amount to invest B to synthesize the option it is only necessary to solve simultaneously the following two equations:

$$\Delta S_u + (1+r)B = c_u$$

$$\Delta S_d + (1+r)B = c_d.$$

Solving these two equations gives

$$\Delta = \frac{c_u - c_d}{S_u - S_d}$$

and

$$B = \frac{1}{1+r}(c_d - \Delta S_d) = \frac{1}{1+r}\left(c_d - \frac{(1+d)S(c_u - c_d)}{(S_u - S_d)}\right).$$

This is illustrated graphically in Figure 1. The payoff at maturity to the call option is illustrated by the thick line. The two possible end values for the stock S_u and S_d are drawn on the horizontal axis. The strike price K has been chosen between the two values S_u and S_d so that in the down state the call option expires valueless and $c_u = 0$. The value of the call in the up state is $c_u = S_u - K$ and this is illustrated on the vertical axis. Also drawn is the line connected the two points (S_d, c_d) and (S_u, c_u). The slope of this line is Δ and the intercept with the vertical axis is $-\Delta S_d = -(1+r)B$. It can be seen that the line is never downward sloping, so $\Delta \geq 0$ and the slope is always less than 45^o, so $\Delta \leq 1$.[1] The intercept with the vertical axis is zero or negative indicating that the portfolio that synthesizes the call option involves borrowing (selling rather than buying the risk-free asset).

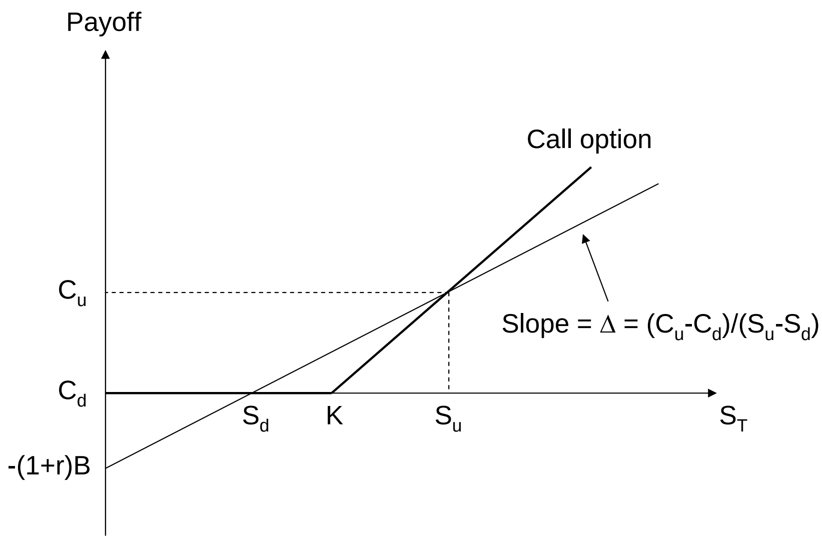

Figure 1: The Δ of a Call Option in the Binomial Model

The cost of synthesizing the option is the cost of the portfolio of Δ units of the shares and investing B in bonds

$$c = \Delta S + B = \frac{pc_u + (1-p)c_d}{(1+r)}$$

where $p = (r-d)/(u-d)$ exactly as before.

[1]It is only equal to zero if $c_u = c_d = 0$ and is only equal to one if $c_u > c_d > 0$.

EXERCISE: Repeat for a put option.

Risk Neutral Pricing

The previous two subsections have derived a simple expression for the price of the call option

$$c = \frac{pc_u + (1-p)c_d}{(1+r)}.$$

In this expression the value of the call option is the *present value* of a weighted average of the call at maturity, either c_u in the up state, or c_d in the down state. It is tempting to interpret p in this equation as a probability. It is important to notice two things about this formula. First p cannot be the probability of an up movement in the stock as the probabilities of stock movements were not used in the calculation of the call value. Thus the value of the call is independent of the probabilities of the up or down movements in the stock price. Second the formula for the call is based on the present value of $pc_u + (1-p)c_d$ and is therefore valued as if this payoff was risk-free. Thus the probability p is a risk-adjusted probability. It is simply the future value of the state price for the up-state.

To give an interpretation to p imagine a situation where all individuals were indifferent to risk. Such as situation is known as a *risk-neutral world.* In such a situation all individuals would agree to value any risky prospect simply by its expected present value. Since the formula for the value of the call is the expected present value when p *is* interpreted as the probability of an upward movement in the stock, ρ is interpreted as a *risk-neutral probability* and this method of valuing the derivative is called *risk neutral valuation*. It is a simple and general principle that can be used for valuing all derivative securities.

To see how it can be used, let's revisit our simple example again. Using the method of risk neutral valuation, the expected value of the stock at the end of the period is $175p + 75(1 - p)$, so the expected present value using the risk-neutral probabilities is

$$\frac{175p + 75(1 - p)}{(1 + 1/4)}.$$

This must equal the actual value of the stock which is 100. This gives one equation in one unknown, p, and solving gives $p = \frac{1}{2}$. To value the call we simple use risk-neutral valuation again. The call at maturity is worth 75 in the up state and 0 in the down state, so the expected value using the risk neutral probabilities is $(1/2)75$ and the present value of this expectation is $(1/2)75/(5/4) = 30$.

Again generalising this method, p is solved from the equation

$$S = \frac{p(1 + u)S + (1 - p)(1 + d)S}{(1 + r)}$$

to give $p = (r - d)/(u - d)$ exactly as before and this value of p can then be used to compute the risk-neutral valuation for the option.

Option Risk

We will now consider the risk of an option and relate it to the standard capital asset pricing model. If we are simply interested in valuing the option, then we can largely ignore this issue. We have just shown in the one-period binomial model that to price an option we only need to know the price of the underlying asset. It is not necessary to know whether the underlying asset is priced fairly relative to other assets or indeed anything about other assets at all. However, if we are interested in forming an investment portfolio that includes options or other derivatives, then the risk and return of that portfolio will depend on the risk and return characteristics of the options. Thus we will need to know how the β of an option relates to the β of the underlying stock.

Risk in the binomial model

We'll use the simple example we used when discussing the one period binomial model. Suppose the price of the underlying stock is 100 and at the end of the period it has either risen to 175 or has fallen to 75. That is $u = 0.75$ and $d = -0.25$, so that the value of the stock after one period in the up state is $100(1+u) = 175$ and the value in the down state is $100(1+d) = 75$. The net growth rate of the value of the stock in the up state is 75%, and the net growth in the value of the stock in the down state is −25%.

We showed that in determining the price of the option, the probability that the stock price rises of falls was irrelevant. Now however suppose that the true probability of the up state is $\pi = 4/5$ and the probability of the down state is $(1-\pi) = 1/5$.[2] The expected rate of return on the stock is therefore $\mu_S = (4/5)75 + (1/5)(-25) = 55\%$ and with an interest rate of 25% the excess over the risk-free rate is $\mu_S - r = 55 - 25 = 30\%$. Given that the price of the call is 30, the rate of return on the call option is $(75-30)/30$ or 150% in the up state and −100% in the down state. Thus the expected rate of return on the call option is $\mu_C = 100\%$ and the excess return over the risk-free rate is $\mu_C - r = 75\%$. Thus the excess return on the call is 2.5 times the excess return on the underlying asset. We know that this extra return will only come at the cost of extra risk.

The standard deviation of the rate of return on the stock is often simply referred to as the stock's *volatility*. The volatility of the stock is

$$\sigma_S = \sqrt{\frac{4}{5}(75-55)^2 + \frac{1}{5}(-25-55)^2} = 40\%$$

[2]Remember from the previous notes that the risk-neutral probability of the up state was $p = 1/2$. The risk-neutral probability is a risk-adjusted probability so it will be adjusted downward from the true probability. This is true in our case as $\pi = 4/5 > 1/2 = p$.

and the standard deviation of the rate of return on the call option, or the call's volatility is

$$\sigma_C = \sqrt{\frac{4}{5}(150 - 100)^2 + \frac{1}{5}(-100 - 100)^2} = 100\%.$$

Thus the standard deviation of the call returns is again 2.5 times the standard deviation of the stock returns.

Now remember that Δ measures the responsiveness of the value of the call to changes in the value of the stock. In our example $\Delta = 3/4$. It is natural to measure this change in percentage terms. That is we define

$$\Omega \equiv \frac{S}{c}\Delta$$

to be the elasticity of the option, that is the percentage change in the call value relative to a given percentage change in the stock value. In our example $S = 100$ and $c = 30$, so that $\Omega = 2.5$.

This is not a coincidence. If π is the probability of the up state and $1-\pi$ is the probability of the down state then expected rate of return on the stock and its volatility are:

$$\mu_S = \pi u + (1-\pi)d \qquad \text{and} \qquad \sigma_S = \sqrt{\pi(1-\pi)}(u-d).$$

Equally the expected rate of return on the call and its volatility are

$$\mu_C = \frac{\pi c_u + (1-\pi)c_d - c}{c} \qquad \text{and} \qquad \sigma_C = \sqrt{\pi(1-\pi)}\frac{c_u - c_d}{c}.$$

Then remembering that Δ is the change in the call price over the change in the stock price, we have

$$\Omega \equiv \frac{S}{c}\Delta = \frac{S}{c}\frac{c_u - c_d}{(u-d)S} = \frac{c_u - c_d}{c(u-d)}.$$

It is therefore clear using the expressions above for σ_C and σ_S that

$$\sigma_C = \Omega\sigma_S.$$

Thus the ratio of the call and stock volatilities is exactly the option elasticity.

Remember that in synthesizing the call option we used the equations

$$\Delta(1+u)S + (1+r)B = c_u$$

$$\Delta(1+d)S + (1+r)B = c_d.$$

to show that the call price is $c = \Delta S + B$. We can therefore substitute for $B = c - \Delta S$ to write these two equations as

$$\Delta(1+u)S - c_u = (1+r)(\Delta S - c)$$

$$\Delta(1+d)S - c_d = (1+r)(\Delta S - c).$$

Multiplying the first equation by π and the second equation by $(1-\pi)$ and adding the resulting two equations gives

$$\pi(\Delta(1+u)S - c_u) + (1-\pi)(\Delta(1+d)S - c_d) = (1+r)(\Delta S - c).$$

Then dividing by c, using the definition of $\Omega = \Delta(S/c)$ and after some rewriting gives

$$\mu_C - r = \Omega(\mu_S - r).$$

That is the risk premium on the call id just Ω times the risk premium on the underlying asset. As we have just shown $\Omega = \sigma_S/\sigma_C$ we can rewrite the expected return on the call option as

$$\mu_C = r + \left(\frac{\mu_S - r}{\sigma_S}\right)\sigma_C.$$

This is of course the standard equation used to express any asset as a portfolio of the stock and the risk-free asset, that we have seen previously in the Asset Pricing and Portfolio Choice courses. As we know the call option can be replicated by a fraction of the stock and selling the risk-free asset this should not be surprising. That is the risk and return of the call will lie somewhere on the risk-return line with intercept of r and slope of $(\mu_S - r)/\sigma_S$. The elasticity Ω gives the position of the call along this line. Rewriting our equation on the excess return again

$$\mu_C = \Omega\mu_S + (1 - \Omega)r.$$

Thus the call can be seen as the portfolio of the stock and the risk-free asset with a weight of Ω attached to the stock. Since we have seen that to synthesize the call requires borrowing the risk-free asset, we have that $\Omega > 1$ and the call volatility is always greater than the stock volatility.

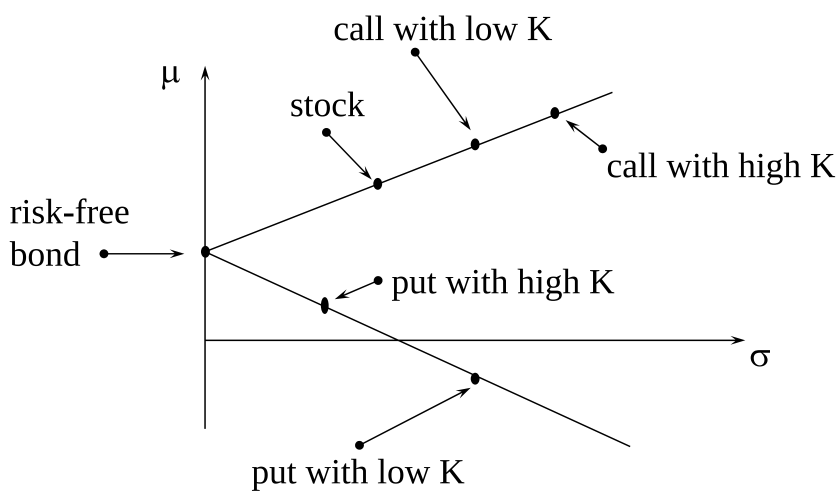

Figure 2: Risk-Return Diagram: (σ, μ)

This is seen in Figure 2. In the diagram the call option is always to the right of the underlying stock as it will have higher volatility and higher expected return than the underlying asset itself. It can also be seen that a call with a higher strike price will be more risky and therefore will lie further out along the risk and return line.

If the same analysis is repeated for a put option it is found that Δ is negative and B is positive. This means that replicating the put option involves selling the underlying asset and buying the risk-free asset (investing). As the underlying asset is sold short a put option will be located on the lower arm of the risk-return diagram. The expected return is reduced because there is a future obligation from the short sale rather than a return although the obligations are still risky and therefore have a positive standard deviation. It can therefore be seen that put options are never of themselves efficient portfolios although since they will be negatively correlated with the underlying asset they can provide good hedging opportunities.

We now consider the β of a call option. The Capital Asset Pricing Model, shows that for any asset, the *excess return* or *risk-premium* satisfies

$$\mu_S - r = \beta_S(\mu_M - r)$$

where μ_M is the expected rate of return on the market portfolio and β_S is the covariance of the stock's rate of return with the market divided by the variance of the rate of return on the market portfolio.

It is therefore simple to combine our excess return equation for the call and the CAPM formula to derive

$$\mu_C - r = \Omega\beta_S(\mu_M - r).$$

It can be shown that $\Omega\beta_S$ is the covariance of the rate of return of the call with the market divided by the variance of the rate of return of the market, so that the beta of the call is $\beta_C = \Omega\beta_S$. The option beta is simply the elasticity time the beta of the underlying asset. Provided that $\beta_S > 0$, since $\Omega > 1$ for a call option, it is the case that $\beta_C > \beta_S$ and the beta for the call is higher than the beta of the underlying asset.[3] This is illustrated in Figure 3 which shows the security market line which plots expected return μ against β. The security market line intercepts the vertical axis at r the rate of return on the risk-free asset which is uncorrelated with the market return (as it is risk-less) and hence has a zero beta. Since $\Omega \geq 1$ the call option has a higher β than the underlying asset and so lies to the right of the stock on the security market line. Put options on the other hand are anti-correlated with the stock, their value goes up as the stock goes down, so they have a negative β and are located to the left of the risk-free asset on the security market line.

Let's return once again to our simple example and suppose that the expected rate of return on the market portfolio is 40%. Then since the excess return on the stock is 30% and the excess return on the market portfolio is $40 - 25 = 15\%$, the beta of the stock is $\beta_S = 2$ and the beta of the option is $\beta_C = \Omega\beta_S = 5$. Since $\beta > 0$ and the expected rate of return on the market portfolio is greater than the risk-free rate, the expected rate of return on the call is also greater than the risk-free rate. Remember that the expected rate of return on the call using the risk-neutral probabilities is equal to the risk-free rate. Thus the risk-neutral probability for the up state is less than the true probability, $p < \pi$. This simply reflects that in adjusting for the risk of the call the probability of the high rate of return is shaded downward to reflect an aversion to risk. In our example the risk-neutral probability is 1/2 and the actual probability of the up state is 4/5.

[3]Often in estimating the CAPM model it is assumed that β_S is constant over time. It would however be totally inappropriate to assume that β_C is constant. The option beta depends on the option elasticity that depends on Δ, S and c and these values will be changing over time.

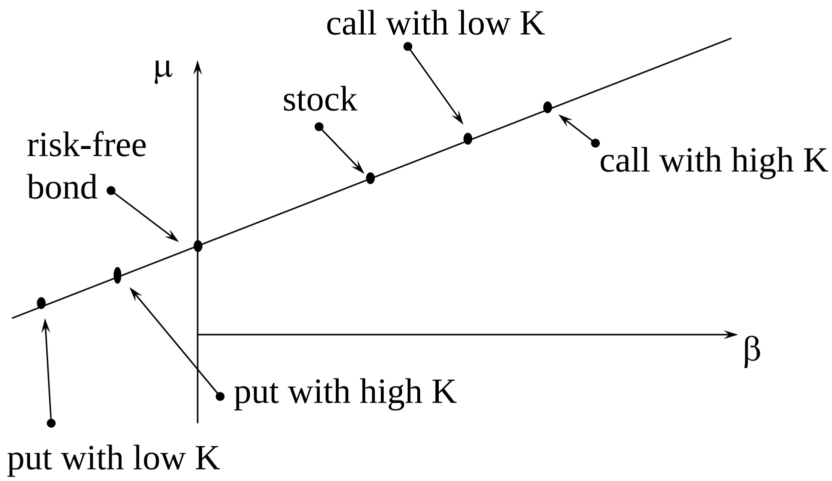

Figure 3: Security Market Line: (β, μ)

Summary

We've considered a one period binomial model where the stock price can go either up or down by factors $(1+u)$ and $(1+d)$. We can the use information on the current stock price S, the risk-free interest rate r and the strike price K to determine the value of the option. We did this in three different ways. First by creating a riskless portfolio of the stock and the option. Second by creating a synthetic option that replicates the payoffs to the option using the underlying stock and the risk-free asset. In both cases we use an arbitrage argument that any two portfolios or assets that deliver the same payoffs must trade at the same price. Thirdly we derived the risk-neutral probability such that all assets and portfolios are valued by their expected present value. This ten allows the option to be evaluated in the same way using these risk-adjusted probabilities.

We've also provided a connection between option pricing and standard models of portfolio choice and the Capital Asset Pricing Model (CAPM). The connection is provided by the option elasticity Ω. We have seen previously that Δ measures how much of the stock to buy to create a riskless hedge or to replicate the option. The option elasticity, Ω is a proportionate measure of Δ, $\Omega = S\Delta/c$. The elasticity Ω therefore measures the proportion of the portfolio placed in the underlying asset in order to replicate the option. It is therefore a portfolio weight rather a quantity of stock bought. The elasticity of the call option will be no less than one since to replicate the call option it will be necessary to borrow, that is go short in the risk-free asset, to fund the acquisition of the underlying asset. The elasticity also can be combined with the *beta* of the stock, β_S to calculate the *beta* of the call option $\beta_C = \Omega\beta_C$. Since $\Omega \geq 1$ the call option will be a more *aggressive* asset that the underlying stock.

The next thing to do is to allow for more than one period in the binomial model to see if the price of the option can still be calculated in a similar and similarly simple fashion.