The Black-Scholes Formula

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We next examine the Black-Scholes formula for European options. The Black-Scholes formula are complex as they are based on the geometric Brownian motion assumption for the underlying asset price. Nevertheless they can be interpreted and are easy to use once understood.

The payoff to a European call option with strike price K at the maturity date T is

$$c(T) = \max[S(T) - K, 0]$$

where $S(T)$ is the price of the underlying asset at the maturity date. At maturity if $S(T) > K$ the option to buy the underlying at K can be exercised and the underlying asset immediately sold for $S(T)$ to give a net payoff of $S(T) - K$. Since the option gives only the right and not the obligation to buy the underlying asset, the option to buy the underlying will not be exercised if doing so would lead to a loss, $S(T) - K < 0$. The Black-Scholes formula for the price of the call option at date $t = 0$ prior to maturity is given by

$$c(0) = S(0)N(d_1) - e^{-rT} K N(d_2)$$

where $N(d)$ is the cumulative probability distribution for a variable that has a standard normal distribution with mean of zero and standard deviation of one. It is the area under the standard normal density function from $-\infty$ to d and therefore gives the probability that a random draw from the standard normal distribution will have a value less than or equal to d. We have therefore that $0 \leq N(d) \leq 1$ with $N(-\infty) = 0, N(0) = \frac{1}{2}$ and $N(+\infty) = 1$. The term r is the continuously compounded risk-free rate of interest and d_1 and

d_2 satisfy

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S(0)}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} = \frac{\ln(\frac{S(0)}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}$$

Here σ^2 is the variance of the continuously compounded rate of return on the underlying asset.

Likewise the payoff to a European put option with strike price K at the maturity date T is

$$p(T) = \max[K - S(T), 0]$$

as the put option gives the right to sell underlying asset at the strike price of K. The Black-Scholes formula for the price of the put option at date $t = 0$ prior to maturity is given by

$$p(0) = c(0) + e^{-rT}K - S(0) = e^{-rT}K(1 - N(d_2)) - S(0)(1 - N(d_1))$$

where d_1 and d_2 are defined above. By the symmetry of the standard normal distribution $N(-d) = (1 - N(d))$ so the formula for the put option is usually written as

$$p(0) = e^{-rT}KN(-d_2) - S(0)N(-d_1).$$

Assumptions Underlying the Black-Scholes Formula

To derive the Black-Scholes formula the following assumptions are required.

1. Markets are perfect: there are no transactions costs or taxes (i.e. there is no bid-ask spread), assets can be short perfectly short sold and are divisible (i.e. we can trade assets in negative and fractional units).
2. There is a risk-free asset and the interest rate on the risk-free asset is unchanging over the life of the option.

3. Trading is *continuous*.

4. The stock price follows a *geometric Brownian motion process*. That is

$$dS = \mu S\,dt + \sigma S\,dz$$

   where S is the stock price, μ is the expected return on the stock (assumed to be unchanging over the life of the option) and σ is the standard deviation of the return on the stock (assumed to be unchanging over the life of the option) and dz is a *Wiener process*.

We will explain a little bit more about assumptions 3 and 4 and how they relate to the Black-Scholes formula.

Continuous Trading

We shall consider the effect of continuous trading on the risk-free asset first. The assumption of continuous trading actually has many advantages mathematically and makes the calculation of rates of return over different periods rather easy.

The Rate of Return

The rate of return is simply the end value less the initial value as a proportion of the initial value. Thus if 100 is invested and at the end value is 120 then the rate of return is $\frac{120-100}{100} = \frac{1}{5}$ or 20%. If the the initial investment is B_{T-1} at time $T-1$ and the end value is B_T after one more period then the rate of return is

$$r = \frac{B_T - B_{T-1}}{B_{T-1}}$$

or equivalently the rate of return r satisfies $B_T = B_{T-1}(1+r)$. We can generalise this to write the rate of return over T periods say when the initial investment is B_0 and write the rate of return as

$$r(T) = \frac{B_T - B_0}{B_0}$$

so that $r(T)$ satisfies $B_T = B_0(1 + r(T))$.

It is important to know the rate of return. However to compare rates of return on different investments with different time horizons it is also important to have a measure of the rate of return per period. One method of making this comparison is to to assume the asset is traded continuously and can be priced by using the *continuously compounded rates of return*. To explain this we first consider compound returns and then show what happens when the compounding is continuous.

Compound Rates of Return

Compound interest rates are calculated by assuming that the principal (initial investment) plus interest is re-invested each period. Compounding might be done annually, semi-annually, quarterly, monthly or even daily. Assuming the re-investment is done after each period, the per-period interest rate r on the investment satisfies

$$(1 + r(T)) = (1 + r)^T.$$

Now consider dividing up each period into n sub-periods each of length Δt. This is illustrated in Figure 1. Then if the compounding is done n times per period, the compound interest rate r satisfies

$$(1 + r(T)) = (1 + \frac{r}{n})^{nT}.$$

For example consider a time period of one-year and suppose an investment of 100 that yields 120 after two years ($T = 2$) has a rate of return $r(2) = 0.2$. If the interest rate is annualised using annual compounding ($n = 1$, $T =$

2), then $r = 0.09544$; with semi-annual compounding ($n = 2$, $T = 2$) the annualised interest rate is $r = 0.09327$; with quarterly compounding ($n = 4$, $T = 2$) the annualised interest rate is $r = 0.0922075$ etc.

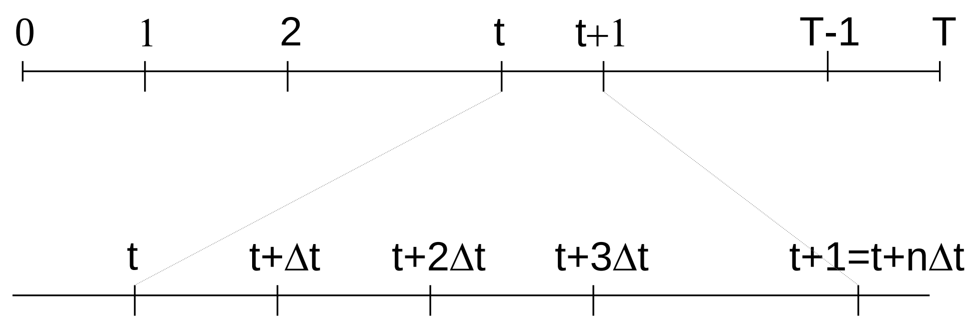

Figure 1: Dividing a time interval n sub-periods

Continuous Compounding

Each period divided into n sub-periods of length Δt times per period and compounding occurs n times per period. As we divide the period into ever more sub-periods the length of time between compounding be denoted by $\Delta t = \frac{1}{n}$ becomes smaller and n grows larger. In this case the compounding factor r satisfies

$$(1 + \frac{r}{n})^{nT} = (1 + r\Delta t)^{\frac{T}{\Delta t}}.$$

Continuous compounding occurs as $\Delta t \to 0$ or equivalently as $n \to 0$. Let $m = \frac{1}{r\Delta t}$, then

$$(1 + r\Delta t)^{\frac{T}{\Delta t}} = (1 + \frac{1}{m})^{mrT} = \left((1 + \frac{1}{m})^{m}\right)^{rT}.$$

As we let the interval between compounding Δt go to zero then $m \to \infty$. The limit of $(1 + \frac{1}{m})^m$ as $m \to \infty$ is well known and is given by the formula

$$\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m = e = 2.71828$$

where $e = 2.7182818$ is the base of the natural logarithm. Thus in the limit the compounding factor is given by

$$(1 + \frac{r}{n})^{nT} = \left((1 + \frac{1}{m})^m\right)^{rT} \to e^{rT}.$$

This gives a simple method for calculating the continuously compounded rate of return r for any given investment. Suppose the initial outlay is B_0 and the value of the portfolio or investment after T periods is B_T. Then since $B_T = B_0(1 + r(T))$ one has

$$\frac{B_T}{B_0} = (1 + r(T)) = e^{rT}$$

so that taking logarithms of both sides

$$\ln\left(\frac{B_T}{B_0}\right) = rT$$

or rewriting we have

$$r = \frac{1}{T}\ln\left(\frac{B_T}{B_0}\right) = \frac{1}{T}(\ln B_T - \ln B_0).$$

This is known as the continuously compounded rate of return.

The Continuously Compounded Rate of Return

The continuously compounded rate of return has the property that longer period rates of return can be computed simply by adding shorter continuously compounded rates of return. This is a very convenient feature which makes using the continuously compounded rates of return especially simple. To see this let r_t denote the continuously compounded rate of return from period t to $t+1$, that is

$$r_t = \ln\left(\frac{B_{t+1}}{B_t}\right)$$

where B_t is the value of the asset at time t. Let $r(T)$ denote the continuously compounded rate of return over the period 0 to T,

$$r(T) = \ln\left(\frac{B_T}{B_0}\right) = \ln B_T - \ln B_0.$$

Suppose that $T = 2$ then we can write this as

$$r(2) = \ln B_2 - \ln B_0 = (\ln B_2 - \ln B_1) + (\ln B_1 - \ln B_0) = r_1 + r_0.$$

Thus the continuously compounded rate of return over two periods is simply the sum of the two period by period returns. In general for any value of T we can write

$$\begin{aligned} r(T) &= (\ln B_T - \ln B_{T-1}) + (\ln B_{T-1} - \ln B_{T-2}) + \ldots + (\ln B_2 - \ln B_1) + (\ln B_1 - \ln B_0) \\ &= r_{T-1} + r_{T-2} + \ldots + r_1 + r_0 = \sum_{t=0}^{T-1} r_t. \end{aligned}$$

Thus the continuously compounded rate of return over time T is simply the sum of the period by period returns. If r_t is constant over time then $r(T) = rT$.

A Differential Equation

It will be useful to think about the value of the risk-free asset as it changes over time. Let r_t denote the rate of return between t and $t+1$. Then over any sub-interval of Δt say between t and $t + \Delta t$, r_t satisfies

$$B_{t+\Delta t} = (1 + r_t \Delta t) B_t.$$

Then taking the limit as $\Delta t \to 0$ we have $B_{t+\Delta t} - B_t \to dB(t)$ where $B(t)$ is the price at time t and $\Delta t \to dt$. Hence we can write

$$dB(t) = r_t B(t) dt$$

or equivalently

$$\frac{dB(t)}{B(t)} = r_t dt.$$

This is a differential equation. If we assume that $r_t = r$ is constant over time ($r_t = r$ for each t) then this equation can be solved at by integration to give the asset price at time T

$$\ln B_T - \ln B_0 == \ln\left(\frac{B_T}{B_0}\right) = rT$$

or

$$B_T = B_0 e^{rT}.$$

Geometric Brownian Motion

We have assumed so far that the rate of return was known so that we were dealing with a risk-free asset. But for most assets the rate of return is uncertain or *stochastic*. As the asset value also changes through time the we say that the asset price follows a **stochastic process**. Fortunately the efficient markets hypothesis provides some strong indication of what properties this stochastic process will have.

A Coin Tossing Example

To examine the form that uncertain returns may take it is useful to think first of a very simple stochastic process. This we have already seen as the binomial model is itself a stochastic process. As an example consider the case of tossing a fair coin where one unit is won if the coin ends up Heads and one unit is lost if the coin ends up Tails. An example of the possible payoffs for a particular sequence of Heads and Tails is illustrated in Figure 2.

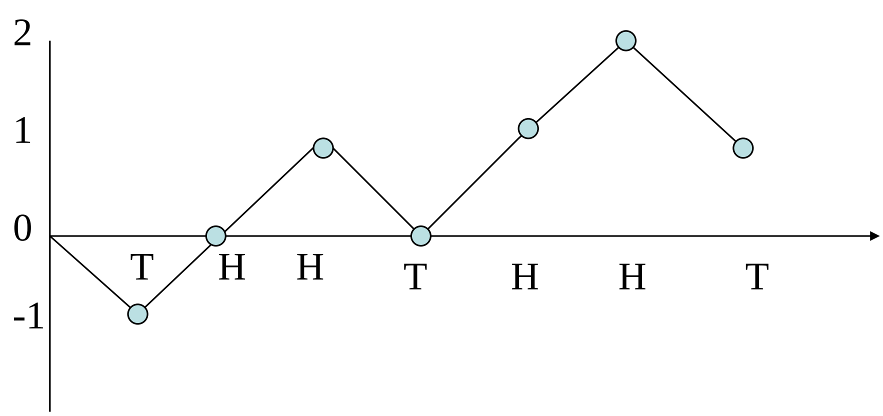

Figure 2: A Coin Tossing Stochastic Process

The important properties of this example are that the distribution of returns are 1) identically distributed at each toss (there is an equal chance of a Head or a Tail); 2) independently distributed (the probability of a Head today is independent of whether there was a Head yesterday); 3) the expected return is the same each period (equal to zero); 4) the variance is constant at each period (equal to one).

There are some important implications to note about this process. First let x_t denote the winnings on the tth toss. We have $x_0 = 0$ and $\mathrm{E}[x_1] = 0$ where $\mathrm{E}[x_t]$ denotes the expected winnings at date t. Then we also have at any date

$$\mathrm{E}[x_{t+1}] = x_t.$$

Any process with this property is said to be a **martingale**. Another important property is that the variance of x is increasing proportionately to the number of tosses. In particular letting σ_t^2 denote the variance of the winnings at the tth toss we have $\sigma_t^2 = t$ or in terms of the standard deviation (the square root of the variance

$$\sigma_t = \sqrt{t}.$$

This is illustrated in Figure 3. Figure 3 shows all possible winnings through four tosses. The variance of winnings is easily calculated at each toss. For example at the second toss the expected winnings are zero so the variance is given by

$$\sigma_2^2 = \frac{1}{4}(2-0)^2 + \frac{1}{2}(0-0)^2 + \frac{1}{4}(-2-0)^2 = 1+1 = 2$$

and so the standard deviation is $\sigma_2 = \sqrt{2}$.

A Stochastic Process for Asset Prices

The efficient markets hypothesis implies that all relevant information is rapidly assimilated into asset prices. Thus asset prices will respond only to new information (news) and since news is essentially unforecastable so to are asset prices. The efficient market hypothesis also implies that it is impossible to consistently make abnormal profits by trading on publicly available information and in particular the past history of asset prices. Thus only the current asset price is relevant in predicting future prices and past prices are irrelevant. This property is know as the **Markov** property for stock prices. If we add a further assumption that the variability of asset prices is roughly constant over time, then the asset price is said to follow a **random walk**. This was true of our coin tossing example above.

Let u_t denote the *random* rate of return from period t to $t+1$. Then

$$S_{t+1} = (1+u_t)S_t.$$

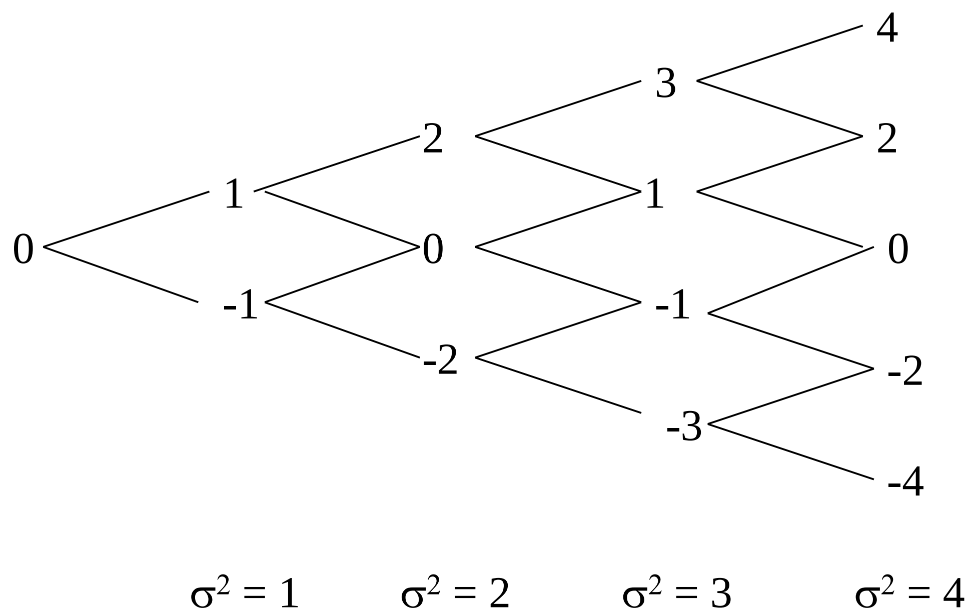

Figure 3: Coin Tossing Example: The Variance is Proportional to Time

The return u_t is now random because the future asset price is unknown.[1] It can be considered as a random *shock* or *disturbance*. Taking natural logarithm of both sides gives

$$\ln(1 + u_t) = \ln S_{t+1} - \ln S_t.$$

Let $\omega_t = \ln(1 + u_t)$. The ω_t is the continuously compounded rate of return from period t to $t + 1$. Since u_t is random ω_t will be random too. We shall however assume

1. ω_t are independently and identically distributed

[1]This was the case in our binomial model where u_t takes on either the value of u in the upstate or d in the down state.

2. ω_t is normally distributed with a mean of ν and variance of σ^2 (both independent of time).[2]

With these assumptions we can see how the stock price is distributed at a future date. Suppose we start from a given value S_0, then

$$\begin{aligned}
\ln S_1 &= \ln S_0 + \ln(1+u_0) \\
\ln S_2 &= \ln S_1 + \ln(1+u_1) = \ln S_0 + \ln(1+u_0) + \ln(1+u_1) \\
\ln S_3 &= \ln S_2 + \ln(1+u_2) = \ln S_0 + \ln(1+u_0) + \ln(1+u_1) + \ln(1+u_2) \\
\vdots &= \vdots \\
\ln S_T &= \ln S_0 + \sum_{t=0}^{T-1} \ln(1+u_t) = \ln S_0 + \sum_{t=0}^{T-1} \omega_t.
\end{aligned}$$

Since ω_t is a random variable which is identically and independently distributed and such that the expected value $\mathrm{E}[\omega_t] = \nu$ and variance $\mathrm{Var}[\omega_t] = \sigma^2$ then taking expectations we have that $\ln S_T$ is normally distributed such that

$$\mathrm{E}[\ln S_T] = \ln S_0 + \nu T$$

and

$$\mathrm{Var}[\ln S_T] = \sigma^2 T.$$

Standard Normal Variable

We have seen that $\ln S_T$ is normally distributed with mean (expected value) of $\ln S_0 + \nu T$ and variance of $\sigma^2 T$. It is useful to transform this to a variable which has a standard normal distribution with mean of zero and standard deviation of one. Such a variable is called a standard normal variable. To

[2] In fact when we take the limit there is no need to assume normality of the distribution as this will be guaranteed by the Central Limit Theorem.

make this transformation, we subtract the mean and divide by the standard deviation (square root of the variance). Thus

$$\frac{\ln S_T - \ln S_0 - \nu T}{\sigma\sqrt{T}} = -\frac{\ln\left(\frac{S_0}{S_T}\right) + \nu T}{\sigma\sqrt{T}}$$

is a standard normal variable. We let $N(x)$ denote the cumulative probability that the standard normal variable is less than or equal to x. A standard normal distribution is drawn in Figure 4. It can be seen that $N(0) = 0.5$ as the normal distribution is symmetric and half the distribution is to the left of the mean value of zero. It also follows from symmetry that if $x > 0$, then $1 - N(x) = N(-x)$. We will use this property later when we look at the Black-Scholes formula.

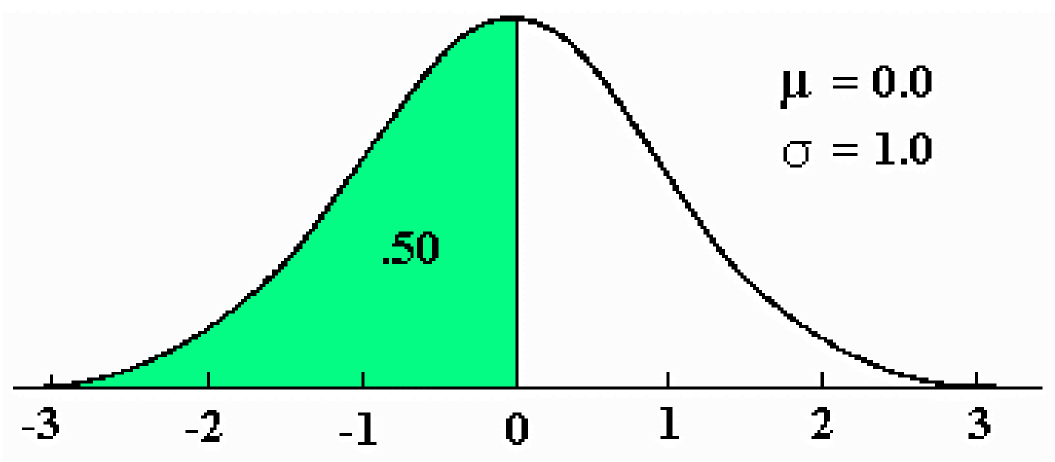

Figure 4: A Standard Normal Distribution. $N(0) = 0.5$.

The Expected Rate of Return

Unfortunately, "the expected rate of return" is a rather ill-defined concept. If we rewrite the above equation for ν we have

$$\nu = \frac{1}{T}\left(\mathrm{E}\left[\ln\left(\frac{S_T}{S_0}\right)\right]\right).$$

This is known as the *expected continuously compounded rate of return*. That is to say we look at the the continuously compounded rates of return and then take expectations. However we can do it another way, which is to compute the expected rates of return and then do our continuous compounding. This would give us

$$\mu = \frac{1}{T}\left(\ln\left(\mathrm{E}\left[\frac{S_T}{S_0}\right]\right)\right).$$

Notice that the difference is that the Expectations operator appears inside the logarithm. The value μ is know as the *logarithm of expected return* or sometimes confusingly as *the expected rate of return*. The values of ν and μ will be different if the asset is uncertain. Fortunately there is a very simple relationship between ν and μ which is given by[3]

$$\mu = \nu + \frac{1}{2}\sigma^2.$$

The reason why μ exceeds ν can be seen because the distribution of asset prices is lognormal.

Lognormal Random Variable

Since the logarithm of the asset price is normally distributed the asset price itself is said to be **lognormally** distributed. In practice when one looks at the empirical evidence asset prices are reasonably closely lognormally distributed.

[3]The equation below always holds approximately and holds exactly in the limit.

We have assumed that that $\omega_t = \ln(1+u_t)$ is normally distributed with an expected value of ν and variance σ^2. But $1+u_t$ is a lognormal variable. Since $1+u_t = e^{\omega_t}$ we might guess that the expected value of u_t is $\mathrm{E}[u_t] = e^{\nu} - 1$. However this would be WRONG. The expected value of u_t is

$$\mathrm{E}[u_t] = e^{\nu+\frac{1}{2}\sigma^2} - 1.$$

The expected value is actually higher than anticipated by half the variance. The reason why can be seen from looking at an example of the lognormal distribution which is drawn in Figure 5. The distribution is skewed and as the variance increases the lognormal distribution will spread out. It cannot spread out too much in a downward direction because the variable is always non-negative. But it can spread out upwards and this tends to increase the mean value. One can likewise show that the expected value of the asset price at time T is

$$\mathrm{E}[S_T] = S_0 e^{(\nu+\frac{1}{2}\sigma^2)T}.$$

Hence

$$\mathrm{E}[S_T] = s_0 e^{\mu T}.$$

That is we "expect" the stock price to grow exponentially at the constant rate μ. However our best forecast would be at a rate of ν and hence we should in some sense always "expect less than the expected".

Arithmetic and geometric rates of return

We now consider μ and ν again. Suppose we have an asset worth 100 and for two successive periods it increases by 20%. Then the value at the end of the first period is 120 and the value at the end of the second period is 144.

Now suppose that instead the asset increases in the first period by 30% and in the second period by 10%. The average or **arithmetic mean** of the return is 20%. However the value of the asset is 130 at the end of first period

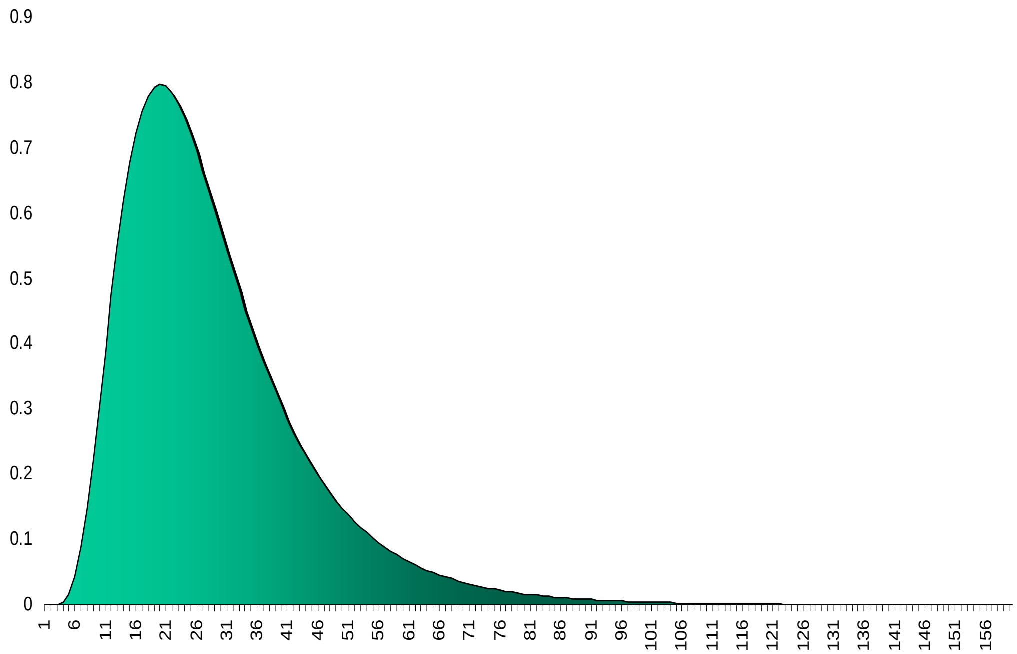

Figure 5: A Lognormal Distribution

and 143 at the end of the second period. The variability of the return has meant that the asset is worth less after two periods even though the average return is the same. We can calculate the equivalent per period return that would give the same value of 143 after two periods if there were no variance in the returns. That is the value ν that satisfies

$$143 = 100(1 + \nu)^2.$$

This value is known as the **geometric mean**. It is another measure of the average return over the two periods. Solving this equation gives the geometric mean as $\nu = 0.195826$ or 19.58% per period[4] which is less than the arithmetic rate of return per period.

[4]The geometric mean of two numbers a and b is $\sqrt{ab}$. Thus strictly speaking $1 + g = 1.195826$ is the geometric mean of 1.1 and 1.3.

There is a simple relationship between the arithmetic mean return, the geometric mean return and the variance of the return. Let $\mu_1 = \mu + \sigma$ be the rate of return in the first period and let $\mu_2 = \mu - \sigma$ be the rate of return in the second period. Here the average rate of return is $\frac{1}{2}(\mu_1 + \mu_2) = \mu$ and the variance of the two rates is σ^2. The geometric rate of return μ satisfies $(1+\nu)^2 = (1+\mu_1)(1+\mu_2)$. Substituting and expanding this gives

$$1 + 2\nu + \nu^2 = (1+\mu+\sigma)(1+\mu-\sigma) = (1+\mu)^2 - \sigma^2 = 1 + 2\mu + \mu^2 - \sigma^2$$

or

$$\nu = \mu - \frac{1}{2}\sigma^2 + \frac{1}{2}(\mu^2 - \nu^2).$$

Since rates of return are typically less than one, the square of the return is even smaller and hence the difference between two squared percentage terms is smaller still. Hence we have the approximation $\nu \approx \mu - \frac{1}{2}\sigma^2$ or

$$\text{geometric mean} \approx \text{arithmetic mean} - \frac{1}{2}\text{variance}.$$

This approximation will be better the smaller are the interest rates and the smaller is the variance. In the example $\mu = 0.2$ and $\sigma = 0.1$ so $\frac{1}{2}\sigma^2 = 0.005$ and $\mu - \frac{1}{2}\sigma^2 = 0.1950$ which is close to the actual geometric mean of 0.1958. Thus the difference between μ and ν is that ν is the geometric rate of return and μ is the arithmetic rate of return. It is quite usual to use the arithmetic rate and therefore to write that the expected value of the logarithm of the stock price satisfies

$$\mathrm{E}[\ln S_T] = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T$$

and

$$\mathrm{Var}[\ln S_T] = \sigma^2 T.$$

The Continuous Trading Limit[5]

This section examines what happens in the limit as the stock price can change continuously and randomly. This section is mathematically more demanding and therefore can largely be ignored. The important result is that with continuous trading the asset price follows a stochastic differential equation given by

$$d\,S(t) = \mu S(t)\,dt + \sigma S(s)\,dz.$$

where dz is a Weiener process which is described in the next section. This process for stock prices is known as *geometric Brownian motion* and shows that the stock price consists of two components. Firstly a deterministic component which shows the stock price growing at a rate of μ and secondly a random or disturbance component with a volatility of σ.

A Wiener Process

Consider a variable z which takes on values at discrete points in time $t = 0, 1, \ldots, T$ and suppose that z evolves according to the following rule:

$$z_{t+1} = z_t + \epsilon; \qquad W_0 \quad \text{fixed}$$

where ϵ is a random drawing from a standardized normal distribution, that is with mean of zero and variance of one. The draws are assumed to be independently distributed. This represents a **random walk** where on average z remains unchanged each period but where the standard deviation of the realized value is one each period. At date $t = 0$, we have $E[z_T] = z_0$ and the variance $Var[z_T] = T$ as the draws are independent.

Now divide the periods into n subperiods each of length Δt. To keep the process equivalent the variance in the shock must also be reduced so that

[5] This section can be considered optional.

the standard deviation is $\sqrt{\Delta t}$. The resulting process is known as a **Wiener process**. The Wiener process has two important properties:

Property 1 The change in z over a small interval of time satisfies:

$$z_{t+h} = z_t + \epsilon\sqrt{\Delta t}.$$

Then as of time $t = 0$, it is still the case that $E[z_T] = z_0$ and the variance $Var[z_T] = T$. This relation may be written

$$\Delta z(t + \Delta t) = \epsilon\sqrt{\Delta t}$$

where $\Delta z(t + \Delta t) = z_{t+\Delta t} - z_t$. This has an expected value of zero and standard deviation of $\sqrt{\Delta t}$.

Property 2 The values of Δz for any two different short intervals of time are independent.

It follows from this that

$$z(T) - z(0) = \sum_{i}^{N} \epsilon_i\sqrt{\Delta t}$$

where $N = T/\Delta t$ is the number of time intervals between 0 and T. Hence we have

$$\mathrm{E}[Z(T)] = z(0)$$

and

$$\mathrm{Var}[z(T)] = N\Delta t = T$$

or the standard deviation of $z(T)$ is $\sqrt{T}$.

Now consider what happens in the limit as $\Delta t \to 0$, that is as the length of the interval becomes an infinitesimal dt. We replace $\Delta z(t + \Delta t)$ by $dz(t)$ which has a mean of zero and standard deviation of dt. This continuous time stochastic process is also known as **Brownian Motion** after its use in physics to describe the motion of particles subject to a large number of small molecular shocks.

This process is easily generalized to allow for a non-zero mean and arbitrary standard deviation. A **generalized Wiener process** for a variable x is defined in terms of $dz(t)$ as follows

$$dx = a\,dt + b\,dz$$

where a and b are constants. This formula for the change in the value of x consists of two components, a deterministic component $a\,dt$ and a stochastic component $b\,dz(t)$. The deterministic component is $dx = a\,dt$ or $\frac{dx}{dt} = a$ which shows that $x = x_0 + at$ so that a is simply the trend term for x. Thus the increase in the value of x over one time period is a. The stochastic component $b\,dz(t)$ adds noise or variability to the path for x. The amount of variability added is b times the Wiener process. Since the Wiener process has a standard deviation of one the generalized process has a standard deviation of b.

The Asset Price Process

Remember that we have

$$\ln S_{t+1} - \ln S_t = \omega_t$$

where ω_t is are independent random variables with mean ν and standard deviation of σ. The continuous time version of this equation is therefore

$$d\,\ln S(t) = \nu\,dt + \sigma\,dz$$

where z is a standard Wiener process. The right-hand-side of the equation is just a random variable that is evolving through time. The term ν is called the **drift** parameter and the standard deviation of the continuously compounded rate of return is $\sqrt{\mathrm{Var}[r(t)]} = \sigma\sqrt{\Delta t}$ and the term σ is referred to as the **volatility** of the asset return.

Ito Calculus

We have written the process in terms of $\ln S(t)$ rather than $S(t)$ itself. This is convenient and shows the connection to the binomial model. However it is usual to think in terms of $S(t)$ itself too. In ordinary calculus we know that

$$d \ln S(t) = \frac{dS(t)}{S(t)}.$$

Thus we might think that $dS(t)/S(t) = \nu\, dt + \sigma\, dz$. But this would be WRONG. The correct version is

$$\frac{dS(t)}{S(t)} = \left(\nu + \frac{1}{2}\sigma^2\right) dt + \sigma\, dz.$$

This is a special case of **Ito's lemma**. Ito's lemma shows that for any process of the form

$$dx = a(x,t)dt + b(x,t)dz$$

then the function $G(x,t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x} a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2(x,t)\right) dt + \frac{\partial G}{\partial x} b(x,t)dz.$$

We'll see how to use Ito's lemma. We have

$$d\ \ln S(t) = \nu\, dt + \sigma\, dz.$$

Then let $\ln S(t) = x(t)$, so $s(T) = G(x,t) = e^x$. Then upon differentiating

$$\frac{\partial G}{\partial x} = e^x = S, \qquad \frac{\partial^2 G}{\partial S^2} = e^x = S, \qquad \frac{\partial G}{\partial t} = 0.$$

Hence using Ito's lemma

$$d\, S(t) = (\nu S(t) + 0 + \frac{1}{2}\sigma^2 S(t))dt + \sigma S(t)\, dz$$

or

$$d\, S(t) = (\nu + \frac{1}{2}\sigma^2) S(t)\, dt + \sigma S(t)\, dz$$

Since $\mu = \nu + \frac{1}{2}\sigma^2$ we can write this as

$$d\,S(t) = \mu S(t)\,dt + \sigma S(s)\,dz.$$

This process is know as geometric Brownian motion as it is the rate of change which is Brownian motion. Thus sometimes the above equation is written as

$$\frac{d\,S(t)}{S(t)} = \mu\,dt + \sigma\,dz.$$

Interpretation of the Black-Scholes Formula

After this rather long digression we return to the Black-Scholes formula $S(T) - K < 0$. The Black-Scholes formula for the price of the call option at date $t = 0$ prior to maturity is given by

$$c(0) = S(0)N(d_1) - e^{-rT}KN(d_2)$$

where d_1 and d_2 satisfy

$$\begin{aligned} d_1 &= \frac{\ln(\frac{S(0)}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} = \frac{\ln(\frac{S(0)}{K}) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}$$

We need to be able to explain and interpret this formula.

Evaluating the Discounted Value of the Future Payoff (Using Risk-Neutral Valuation)

Rewrite the Black-Scholes formula as

$$c(0) = e^{-rT}\left(S(0)e^{rT}N(d_1) - KN(d_2)\right).$$

The formula can be interpreted as follows. If the call option is exercised at the maturity date then the holder gets the stock worth $S(T)$ but has to pay the strike price K. But this exchange takes place only if the call finishes in the money. Thus $S(0)e^{rT}N(d_1)$ represents the future value of the underlying asset conditional on the end stock value $S(T)$ being greater than the strike price K. The second term in the brackets $KN(d_2)$ is the present value of the known payment K times the probability that the strike price will be paid $N(d_2)$. The terms inside the brackets are discounted by the risk-free rate r to bring the payments into present value terms. Thus the evaluation inside the brackets is made using the risk-neutral or martingale probabilities and $N(d_2)$ represents the probability that the call finishes in the money recall in a risk neutral world.

Remember that in a risk-neutral world all assets earn the risk-free rate. we are assuming the the logarithm of the stock price is normally distributed. Thus ν the expected continuously compounded rate of return in a risk neutral world is equal to $r - \frac{1}{2}\sigma^2$ where the variance is deducted to calculate the certainty equivalent rate of return. Therefore at time T ln $S(T)$ is normally distributed with a mean value of ln $S(0)+(r-\frac{1}{2}\sigma^2)T$ and a standard deviation of $\sigma\sqrt{T}$. Thus

$$\frac{\ln\, S(T) - (\ln\, S(0)) + (r - \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}$$

is a standard normal variable. The probability that $S(T) < K$ is therefore given by $N(-d_2)$ and the probability that $S(T) > K$ is given by $1-N(-d_2) = N(d_2)$.

It is more complicated to show that $S(0)e^{rT}N(d_1)$ is the future value of underlying asset in a risk-neutral world conditional on $S(T) > K$ but a proof can be found in more advanced textbooks.

The Replicating Portfolio

The formula also has another useful interpretation. From our analysis of the binomial model we know that the value of the call is

$$c(0) = S(0)\Delta - B$$

where Δ is the amount of the underlying asset bought and B is the amount of money borrowed needed to synthesize the call option. From the formula therefore $N(d_1)$ is the hedge parameter indicating the number of shares bought and $e^{-rT}KN(d_2)$ indicates the amount of cash borrowed to part finance the share purchase. Since $0 \leq N(d) \leq 1$ the formula shows that the replicating portfolio consists of a fraction of the underlying asset and a positive amount of cash borrowed. The Δ of this formula is also found by partially differentiating the call price formula

$$\frac{\partial c(0)}{\partial S(0)} = \Delta.$$

It is the slope of the curve relating the option price with the price of the underlying asset. The cost of buying Δ units of the stock and writing one call option is $S(0)\Delta - c(0)$. This portfolio is said to be **delta neutral** as a small change in the stock price will not affect the value of the portfolio.

Boundary Conditions

A boundary condition tells us the price of the put or call at some extreme value. Thus for example if we have a call option which is just about to expire, then the value of the option must just be its intrinsic value $\max[S(T)-K, 0]$, as the option has no time value left. If the Black-Scholes formula is correct it must reduce to the intrinsic value when the time to maturity left is very small. That is $c(0) \to \max[S(0) - K, 0]$ as $T \to 0$.[6] We will chck that the boundary conditions are indeed satisfied.

Boundary Conditions for a Call Option

We shall consider the boundary conditions for the call option. Consider first the boundary condition for the call at expiration when $T = 0$. To do this consider the formula for the call option as $T \to 0$, that is as the time until maturity goes to zero. At maturity $c(T) = \max[S(T) - K, 0]$ so we need to show that as $T \to 0$ the formula converges to $c(0) = \max[S(0) - K, 0]$. If $S(0) > K$ then $\ln(\frac{S(0)}{K}) > 0$ so that as $T \to 0$, d_1 and $d_2 \to +\infty$. Thus $N(d_1)$ and $N(d_2) \to 1$. Since $e^{-rT} \to 1$ as $T \to 0$ we have that $c(0) \to S(0) - K$ if $S(0) > K$. Alternatively if $S(0) < K$ then $\ln(\frac{S(0)}{K}) < 0$ so that as $T \to 0$, d_1 and $d_2 \to -\infty$ and hence $N(d_1)$ and $N(d_2) \to 0$. Thus $c(0) \to 0$ if $S(0) < K$. This is precisely as expected. If the option is in the money at maturity, $S(0) > K$, it is exercised for a profit of $S(0) - K$ and if it is out of the money, $S(0) < K$, the option expires unexercised and valueless.

As another example consider what happens as $\sigma \to 0$. In this case the underlying asset becomes riskless so grows at the constant rate of r. Thus the future value of the stock is $S(T) = e^{rT}S(0)$ and the payoff to the call option at maturity is $\max[e^{rT}S(0) - K, 0]$. Thus the value of the call at date $t = 0$ is $\max[S(0) - e^{-rT}K, 0]$. To see this from the formula first consider the case where $S(0) - e^{-rT}K > 0$ or $\ln(\frac{S(0)}{K}) + rT > 0$. Then as $\sigma \to 0$, d_1 and $d_2 \to +\infty$ and hence $N(d_1)$ and $N(d_2) \to 1$. Thus $c(0) \to S(0) - e^{-rT}K$. Likewise when $S(0) - e^{-rT}K < 0$ or $\ln(\frac{S(0)}{K}) + rT < 0$ then d_1 and $d_2 \to -\infty$ as $\sigma \to 0$. Hence $N(d_1)$ and $N(d_2) \to 0$ and so $c(0) \to 0$. Thus combining both conditions $c(0) \to \max[e^{rT}S(0) - K, 0]$ as $\sigma \to 0$.

[6] In fact the Black-Scholes formula is usually derived by computing the process the call option price must follow and imposing the boundary conditions.

Boundary Conditions for a Put Option

For a put option $p(T) = \max[K - S(T), 0]$ so we need to show that as $T \to 0$ the formula converges to $p(0) = \max[K - S(0), 0]$. If $S(0) > K$ then $\ln(\frac{S(0)}{K}) > 0$ so that as $T \to 0$, d_1 and $d_2 \to +\infty$. Thus $N(-d_1)$ and $N(-d_2) \to 0$. Since $e^{-rT} \to 1$ as $T \to 0$ we have that $p(0) \to 0$ if $S(0) > K$. Alternatively if $S(0) < K$ then $\ln(\frac{S(0)}{K}) < 0$ so that as $T \to 0$, d_1 and $d_2 \to -\infty$ and hence $N(-d_1)$ and $N(-d_2) \to 1$. Since $e^{-rT} \to 1$ as $T \to 0$ we have that $p(0) \to K - S(0)$ if $S(0) < K$ This is precisely as expected. If the option is in the money at now (at the maturity date $T = 0$) and $S(0) < K$, it is exercised for a profit of $K - S(0)$ and if it is out of the money, $S(0) > K$, the option expires unexercised and valueless.

Now consider what happens to the put option as $\sigma \to 0$. In this case the underlying asset becomes riskless so grows at the constant rate of r. Thus the future value of the stock is $S(T) = e^{rT}S(0)$ and the payoff to the put option at maturity is $\max[K - e^{rT}S(0), 0]$. Thus the value of the put at date $t = 0$ is $\max[e^{-rT}K - S(0), 0]$ (that is discounting by the factor e^{-rT}. To see this from the formula first consider the case where $e^{-rT}K - S(0) > 0$ or taking logs and rearranging $\ln(\frac{S(0)}{K}) + rT < 0$. Then as $\sigma \to 0$, d_1 and $d_2 \to -\infty$ and hence $N(-d_1)$ and $N(-d_2) \to 1$. Thus $p(0) \to e^{-rT}K - S(0)$. Likewise when $e^{-rT}K - S(0) < 0$ or $\ln(\frac{S(0)}{K}) + rT > 0$ then d_1 and $d_2 \to +\infty$ as $\sigma \to 0$. Hence $N(-d_1)$ and $N(-d_2) \to 0$ and so $p(0) \to 0$. Thus combining both conditions $p(0) \to \max[e^{rT}K - S(0), 0]$ as $\sigma \to 0$.