Options

These notes describe the payoffs to **European** and **American** put and call options—the so-called *plain vanilla* options. We consider the payoffs to these options for holders and writers and describe both the time vale and intrinsic value of an option. We explain why options are traded and examine some of the properties of put and call option prices. We shall show that longer dated options typically have higher prices and that call prices are higher when the **strike price** is lower and that put prices are higher when the strike price is higher.

1 Options

Options are **derivative** assets. That is their payoffs are derived from the payoff on some other **underlying asset**. The underlying asset may be an equity, an index, a bond or indeed any other type of asset. Options themselves are classified by their **type**, **class** and their **series**. The most important distinction of types is between put options and call options. A put option gives the owner the right to sell the underlying asset at specified dates at a specified price. A call option gives the owner the right to buy at specified dates at a specified price. Options are different from forward/futures contracts as a put (call) option gives the owner right to sell (buy) the underlying asset but not an obligation. The right to buy or sell need not be exercised. There is also a last date on which the right may be exercised and this is known as the **expiration date** or **maturity date**. The specified price is called the **strike price** or **exercise price**.

The most commonly traded option is the **American** type which may be exercised at any trading date prior to or at the expiration date. There are also options of the **European** type which may only be exercised at the expiration date. Thus an American put and a European put are different types of option. Another type is the **Bermudan** option which is half way between the European and American option as it can only be exercised on a limited number of pre-specified days prior to maturity. We shall focus mainly on European options as there is an important parity condition between the prices of European put and call options and we shall explain how the prices of American options differ from their European counterparts.

The owner or buyer of an option is often called the **holder** and the seller of an option is termed the **writer**. Thus at date of sale the holder pays the writer the price of the option. For a call option there may be a subsequent transaction at the expiration date where the writer provides the underlying asset to the holder in exchange for the agreed strike price. This subsequent transaction is made at the instigation of the holder and will only be undertaken if it is in the holder's interest. The holder's right to buy the stock at or prior to the expiration date need not be exercised. For a put option, the subsequent transaction is that the holder provides the underlying asset in return for the agreed strike price from the writer.

The difference between American and European options is simply what choices the holder has prior to the expiration date. The holder of the European option has, at any point prior to the expiration date, the choice either to sell the option for the prevailing market price or to retain it. The holder of an American option has, at any point prior to the expiration date, three choices: either to sell the option for the prevailing market price, to retain it or to exercise it. The later case is said to involve *early* exercise. This terminology is not meant to imply that there is anything sub-optimal about exercising before the expiration date. Indeed we will give below examples where the American option will be exercised early.

A class of option is all options of a particular type on the same underlying asset. Thus all American put options on IBM stock form one class. European call options on the FTSE 100 index form another class. A class will involve a variety of different strike prices and maturity dates. Most options are exchange traded and the exchange will specify a range of maturity dates, usually every two months, and a range of strike prices, usually centered around the prevailing market price. An option series is simply the options of a given class with the same strike price and maturity date.

Payoff at Maturity

Let K be the exercise price and S_T the price of the underlying stock at the maturity date. Then the payoff to a put option is

$$p_T = \begin{cases} K - S_T & \text{if } S_T \leq K \\ 0 & \text{if } S_T > K. \end{cases}$$

or more simply $p_T = \max[K - S_T, 0]$. If $S_T < K$, then the put is said to **finish in-the-money** and the option will be exercised. The holder of this option will buy the underlying stock at a price of S_T and exercise their right to sell it to the writer at the strike price of K, to make a profit of $K - S_T$. If $S_T > K$, the option is said to **finish out-of-the-money** and exercising the right to sell the underlying asset would result in a loss. So the right to sell won't be exercised. The payoff will be zero in this case. Hence with an option the payoff is never negative. Assets with non-negative payoffs are know as **limited liability** assets.

Suppose that you hold a put option on a stock PDQ. The exercise price is 1000 and the expiration date is in four weeks. The current stock price of PDQ is 1109. If the stock is still at 1109 in four weeks time, you we let the option expire without exercising your right to sell at 1000. If you exercised your right to sell you would have to deliver the stock, which would cost you 1109, and you would receive only 1000 in return. Clearly exercising the option to sell would result in a loss. You will not exercise and your payoff is $\max[K - S_T, 0] = \max[1000 - 1109, 0] = 0$. However, if PDQ does badly and the stock price falls to 900 after four weeks, you as holder of the put option will do well. In that case you can exercise your right to sell, buying the stock at the reduced price of 900 and selling it to the holder at 1000. The payoff at maturity is $\max[K - S_T, 0] = \max[1000 - 900, 0] = 100$ per unit.

Equally the payoff to the holder of a call option is

$$c_T = \begin{cases} 0 & \text{if } S_T \leq K \\ S_T - K & \text{if } S_T > K. \end{cases}$$

or more simply $c_T = \max[0, S_T - K]$. In this case the call finishes in the money if $S_T > K$ as the option holder can exercise the right to buy the underlying asset at a price of K when it is worth the greater amount S_T. If $S_T \leq K$, the call option finishes out of the money and the right to buy will go unexercised.

The writer of the call option has exactly the reverse of the payoff to the holder. The writer of the call has a payoff of $-c_T = -\max[0, S_T - K] = \min[0, K - S_T]$. When the option is exercised the writer will have to deliver a stock worth S_T and receives a payment of K from the holder. Since $S_T > K$, the writer makes a loss of $K - S_T$. Likewise the payoff at maturity to the writer of a put option is $-p_T = -\max[K - S_T, 0] = \min[S_T - K, 0]$.

The **intrinsic value** or **parity value** of an option at time t is the payoff to the option if the current date were the maturity date. Thus the intrinsic value of the call option at time t is $\max[0, S_t - K]$ where S_t is the current price of the underlying asset and the intrinsic value of a put option is $\max[K - S_t, 0]$.

An American option will always trade at a price at or above its intrinsic value, since with an American option it is always possible to exercise the option now and realise the intrinsic value. The difference between the price of an option an its intrinsic value is known as the **premium** or **time value** of the option. Thus the price of an option is the sum of its intrinsic value plus its premium

$$\text{price of option} = \text{intrinsic value} + \text{time value}.$$

If an option is initially set up at date 0 when the stock price is S_0 and with the strike price set such that $K = S_0$ then the intrinsic value of the option is 0 and the premium and price are equivalent. Many options were historically set up with the strike price equal to the prevailing price of the underlying asset (or *at-the-money* as it is called) and this accounts for why the option price is sometimes referred to as the option premium. We will use the term time value rather than premium to avoid confusion.

Why trade options?

It is easy to see why it might be desirable to trade options. Suppose you buy a call option with an exercise price equal to the current stock price. If the stock goes up in value, you can still buy at the low current price and sell at the new higher price. And if you are unlucky and the price falls, then you simply don't exercise the option. All you lose is the price you paid for the option in the first place. You only buy the stock when the price has gone up. You get all the upside benefits and eliminate the downside risk. You don't have to risk buying the stock in that hope that the price will rise. You simply buy the option which costs a fraction of the price of the stock itself.

Because you only pay for the stock at maturity, buying a call option is equivalent to borrowing most of the money to buy the stock and repaying only if the bet goes well. Thus a call option is a highly **levered** or **geared** position in the stock and thus with a return that is higher but also highly risky.

To see this consider the following simple example. The current price of the stock is 100 and a call option on the stock with a strike price of 100 costs 30. Suppose that the stock price at maturity has risen to 175. The rate of return on the stock is a handsome 75%. However the return on the call option is greater still. If the stock price rises to 175 the call option can be exercised to make a profit of $175 - 10 = 75$. Since the call costs 30 the rate of return is $\frac{75}{30} - 1 = 1.5$ or 150%. The high leverage makes possible a high rate of return on the relatively small investment. On the other hand suppose the stock finishes out of the money at 75. Owning the stock results in a rate of return of -25%. The call option however, expires valueless, so the rate of return on the investment in the call option is -100%. Not such a good prospect.

Suppose these are the only two possible values for the stock at maturity and suppose that π is the probability the stock price rises and $(1 - \pi)$ is the probability it falls. Suppose that the interest rate over the period is 25%. The risk premium on the stock is

$$rp_S = 75\pi + (-25)(1 - \pi) - 25 = 100\pi - 50$$

and the risk premium on the call option is

$$rp_C = 150\pi + (-100)(1 - \pi) - 25 = 250\pi - 100.$$

It is clear that the risk premium on the call is 2.5 times the risk premium for the stock irrespective of the value of π. This ratio is known as the **option elasticity** and it can be shown that for a call option it is always greater than one. Thus the call option is always riskier than the underlying stock. The high risk means that options are good and cheap ways to hedge risk as we shall see later on.

2 Properties of Puts and Calls

Table 1 gives the prices of options on two stocks, Tesco and Rolls Royce given in the *Financial Times* on Tuesday March 6th 2000. The starred number in the left column gives the closing stock price on the previous day and the paper reports the prices for puts and calls of three maturities, May, August and November, for strike prices on either side of the closing price.

		···	**Calls**	···	···	**Puts**	···
Option		**May**	**Aug**	**Nov**	May	Aug	Nov
Tesco	160	**$22\frac{1}{2}$**	**27**	**31**	7	$10\frac{1}{2}$	$14\frac{1}{2}$
(*176)	180	**11**	**17**	**23**	16	$19\frac{1}{2}$	25
Rolls Royce	180	**22**	**27**	**$30\frac{1}{2}$**	8	$12\frac{1}{2}$	16
(*$194\frac{1}{2}$)	200	**11**	**$17\frac{1}{2}$**	**$21\frac{1}{2}$**	18	23	$26\frac{1}{2}$

Table 1: Call and Put Prices

Three things are immediately obvious from this table. First the option prices increase with the maturity date for any given strike price. Second the price of calls falls with the strike price for any given maturity date and third the price of puts rises with the strike price with any given maturity date.

For a given strike price, the table shows that the price of calls increases with the date to maturity. That is $C_t(T_2) \geq C_t(T_1)$ for $T_2 > T_1$ where $C_t(T_1)$ is the price of an American call option at date t that matures at date T_1 and $C_t(T_2)$ is the an option of the same type and class and with the same strike price but with a longer maturity.[1] From the table we can see that August calls on Tesco stock with a strike price of 160 trade at 27 but the longer maturity November calls trade at 31. There would be a simple arbitrage opportunity if $C_t(T_1) > C_t(T_2)$ for $T_2 > T_1$. Suppose that the prices were reversed and the November call on Tesco trades at 27 and the August call trades at 31. Then buying the lower priced November call and writing the August call yields a net inflow of 4 today. Either the August call expires or is exercised prior to maturity. In either case its value is $\max[0, S_{t'} - K]$ where t' is the maturity date in August or some time prior to the maturity date when the call is exercised. The value of the position is $C_{t'}(Nov) - \max[0, S_{t'} - K]$. If this is positive then sell the November call at date t' to make a profit. Otherwise exercise the November call. Exercising the call yields the same

[1] We will use a small case $c_t(T)$ to denote the price at date t of a European call option with maturity date T and the large case $C_t(T)$ for an equivalent American option.

value of $\max[0, S_{t'} - K]$ no matter what the maturity date. So there is a completely offsetting gain from the bought November call and the written August call. In either case an arbitrage profit has been made. Note that this does not say anything about how the call value changes over time until maturity. It only compares prices at a particular date of options with different expiration dates. Neither does the argument necessarily work for European options which cannot be exercised early. However, as a empirical matter European options do demonstrate the same pattern of prices rising with date to maturity.

As a short digression it is also true that the price of a European call option on a non-dividend paying stock will increase with the length of time until maturity. This is because a European call option on a non-dividend paying stock has a positive time value. That is the price of a European call option cannot be less than its intrinsic value. Remember that the lower bound for a European call option[2] is $S_t - K(1+r)^{-1}$. Since the call cannot command a negative price we have for the arbitrage lower bounds for a European call option is[3]

$$c_t \geq \max\left[S_t - \frac{K}{(1+r)}, 0\right].$$

Since the strike price on the RHS is discounted we have $\frac{K}{(1+r)} < K$, so that

$$c_t \geq \max\left[S_t - \frac{K}{(1+r)}, 0\right] > \max[S_t - K, 0]$$

for $t < T$ and $r > 0$, so the European call has value greater than its intrinsic value at any date prior to maturity. Remember that an American option cannot be less valuable that a European option, $C_t \geq c_t$ as the American

[2]This lower bound is derived by buying the call, short selling the underlying and lending the discounted value of the strike price. The payoff to this strategy is 0 if $S_T > K$ and $K - S_T > 0$ if $S_T < K$.

[3]I simplify notation by writing $(1+r)$ instead of $(1+r)^{-(T-t)}$. Thus in this case r should always be interpreted as the risk-free interest rate that applies for borrowing/lending between the current date t and the maturity date T rather than an annual equivalent.

option has all the same opportunities to exercise for profit as the European option, and more besides. It therefore follows that if the option of the American type, it will always be better to sell the option rather than exercise it early (for a profit of $S_t - K$) as $C_t \geq c_t > S_t - K$. Since it is never optimal to exercise the American option early, the opportunities to exercise have no value and European and American call options on non-dividend paying stock must trade at the same price, $C_t = c_t$. Thus the argument that American options increase in value with time to maturity also applies to European options on non-dividend paying stock.

Returning to the table it is equally clear is that a call with a lower strike price must command a higher price. Thus $C_t(K_1) \geq C_t(K_2)$ if $K_2 > K_1$. This can be seen from Table 1. For example, May calls on Rolls-Royce with a strike price of 200 are worth 11 but the May calls with the lower strike price of 180 are worth 22. This is simply because the call gives the holder the option to buy and the lower the exercise price at which the stock can be bought the more valuable is the option. If this were not true and $C_t(K_1) < C_t(K_2)$ for $K_2 > K_1$, then there is an arbitrage opportunity and the appropriate response is a **bull spread**. The payoffs can be seen in Table 2. This strategy buys the call with the lower strike price of K_1 and writes the call with the higher strike price of K_2 leading to cash inflow now of $C_t(K_2) - C_t(K_1) > 0$. In each case the payoff at maturity is non-negative. Thus for European options $c_t(K_1) \geq c_t(K_2)$ for $K_2 > K_1$. Equally suppose the options are of the American type, then if the written option is exercised early, the holder of that option will pay us K_2 for the stock. If we exercise the bought option immediately, the stock can be bought for K_1 and then sold for K_2 leading to a profit of $K_2 - K_1$. Hence we have for American options $C_t(K_1) \geq C_t(K_2)$ for $K_2 > K_1$.

Position	$S_T < K_1 < K_2$	$K_1 < S_T < K_2$	$K_1 < K_2 < S_T$
Long Call	0	$S_T - K_1$	$S_T - K_1$
Short Call	0	0	$K_2 - S_T$
Overall	0	$S_T - K_1 > 0$	$K_2 - K_1 > 0$

Table 2: Bull Spread

3 Summary

We have seen how the payoffs to put and call options depend on the underlying assets and how the prices of puts and calls depend on the length of time to maturity and strike price. Our next step is to see how the option price can be exactly determined. To do this we need to make assumptions about how the price of the underlying is changing. This we will do in the next notes on the binomial model.