KEELE UNIVERSITY

HIGHER DEGREE EXAMINATIONS, 2007

FRIDAY 18th MAY, 16.00-18.00

MSc FINANCE AND INFORMATION TECHNOLOGY

MSc FINANCE AND MANAGEMENT

FIN 40008

FINANCIAL INSTRUMENTS

Candidates should attempt FOUR questions from Section A, and TWO questions from Section B. Section A is worth 60 marks and Section B is worth 40 marks.

When presenting numerical results, please give a complete step-by-step presentation of your derivations. All mathematical derivations should be accompanied by brief explanatory remarks and interpretations.

The use of hand-held, battery-operated electronic calculators will be permitted, subject to the regulations governing their use which are displayed outside the examination room. The type of calculator must be specified on the cover sheet of your answer book.

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FIN 40008: FINANCIAL INSTRUMENTS

Candidates should attempt FOUR questions from Section A, and TWO questions from Section B. Section A is worth 60 marks and Section B is worth 40 marks.

SECTION A

Answer FOUR questions in this section. Each question carries 15 marks.

1. Suppose you buy a call option at price C on a underlying asset with current price S_0. The option expires at date T and has a strike price of K.

   (a) What is the intrinsic and the time value of the option?

   (b) Describe the payoff you have at the expiration date T.

   (c) Explain why it can be expected that $C \leq S_0$.

2. Outline two of the following measures of risk and explain how they relate to each other:

   Conditional Value at Risk

   Roy's safety-first index

   The Domar-Musgrave index

   Value at Risk

3. Consider the one period binomial model with parameters $U = 2$ and $D = 1/2$ (i.e., where the stock increases by a factor of 2 or decreases by a factor of 1/2). Suppose that the stock price is initially 150 and that the risk-free interest rate is 50% per period.

   (a) Consider a call option with a strike price of 165 and expiry in one period. Construct the Δ-hedge needed to create a risk-less portfolio of the stock and a written call. Verify that the payoffs to the strategy are risk-free.

   (b) Calculate the cost of the portfolio in part (a) and hence value the call option.

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4. Consider two European call options on an underlying stock with strike prices K_1 and K_2 where $K_2 > K_1$. Both options expire on the same date. Let the price of these two call options be C_{K_1} and C_{K_2}. Suppose you form a bull spread by buying the option with the lower strike price K_1 and selling the call option with the higher strike price K_2.

   (a) Describe the payoff you have at the expiration date.

   (b) Using your answer to part (a) explain why $C_{K_1} > C_{K_2}$.

   (c) Now consider borrowing the discounted value of the difference in the strike prices in addition to the bull spread. What does the payoff to this new portfolio imply about the relationship between the prices C_{K_1} and C_{K_2}?

5. Describe the four different varieties of barrier options. How do the payoffs of the barrier options relate to the payoff of the corresponding plain vanilla option?

6. What is meant by normal backwardation and contango? Why is normal backwardation normal?

7. Use the put-call parity condition, $C - P = S_0 - Ke^{-rT}$, for European options on non-dividend paying stock to derive the relationship between:

   (a) The Δ of a call option and the Δ of a put option.

   (b) The Γ of a call option and the Γ of a put option.

   (c) The Θ of a call option and the Θ of a put option.

   (d) Briefly explain what each of Δ, Γ and Θ measure.

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SECTION B

Answer TWO questions in this section. Each question carries 20 marks.

8. An Ito process is a stochastic process of the form

$$dx = a(x,t)dt + b(x,t)dz$$

where t is time and z is a Wiener process. Ito's lemma then shows that any function $G(x,t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right)dt + \frac{\partial G}{\partial x}b(x,t)dz.$$

It is often assumed that stock prices follow a special type of an Ito process, the geometric Brownian motion process described by

$$dS = \mu S dt + \sigma S dz$$

where S is the stock price, μ is the expected return and σ is the volatility.

(*a*) Use Ito's lemma, the forward price equation $F(t) = S(t)e^{r(T-t)}$ and the assumption of geometric Brownian motion for the stock price to show that the forward price is also geometric Brownian motion with expected return $\mu - r$.

(*b*) Assume that a derivative and the underlying can be traded to construct a risk-free portfolio. Hence use Ito's lemma again to derive the Black-Scholes-Merton partial differential equation

$$rf = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2.$$

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9. A European put option with a strike price of 250 matures in one year. Divide the one-year period into two six-month intervals. The up (U) and down (D) factors for the underlying asset are given by:

$$U = e^{(r-\sigma^2/2)h+\sigma\sqrt{h}}$$
$$D = e^{(r-\sigma^2/2)h-\sigma\sqrt{h}}$$

where r is the continuously compounded risk-free rate of interest, σ is the volatility of the continuously compounded rate of return of the underlying asset and h is the time interval. The risk-free interest rate is 8% and the volatility of the underlying asset is 45%. Using the values $r = 0.08$, $\sigma = 0.45$ and $h = 0.5$ gives:

$$U = 1.36012$$
$$D = 0.71977$$
$$R = e^{rh} = 1.04081.$$

The current price of the underlying asset is 250.

(a) Calculate the binomial tree for the price of the underlying asset and calculate the risk-neutral probability for an up state.

(b) Determine the price of the put along the tree using the risk-neutral probabilities.

(c) What are the put prices if the option is American rather than European?

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10. Adopting standard notation the Black-Scholes formula for the price of a European call option is

$$C_0 = S_0 N(d_1) - e^{-rT} K N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and $N(x)$ is the cumulative probability distribution for a standard normally distributed random variable.

(*a*) Give a brief intuitive explanation for the formula.

(*b*) Use the put-call parity condition to derive the Black-Scholes formula for the equivalent put option.

(*c*) The delta of the call option is $N(d_1)$. Explain the significance of delta in option pricing. How does delta depend upon S_0?

(*d*) Suppose $S_0 = 189$, $K = 190$, $r = 0.05$, $T = 0.25$ and $\sigma = 0.44$. This gives $N(d_1) = 0.6512$ and $N(d_2) = 0.5669$.

(i) Calculate the price of the call option C_0.

(ii) If you buy $N(d_1)$ units of the stock, how much would you have to borrow to have a position equivalent to the call option itself?

(iii) Briefly explain why a levered position in the stock is equivalent to holding the call option.

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11. Assume that the log stock price is a generalised Wiener process

$$d \ln S = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$$

where z is a Wiener process, μ is the expected return and σ is the standard deviation of the return.

(*a*) What are the two main properties of a Wiener process?

(*b*) Show that

$$\ln S_T \sim \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right)$$

where S_T is the stock price at date T and S_0 is the initial stock price.

(*c*) Let $\eta = (1/T)(\ln S_T - \ln S_0)$. Show that

$$\eta \sim \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}}\right).$$

(*d*) Using the fact that $\mathrm{E}[S_T] = S_0 e^{\mu T}$, explain the relationship between μ and η.

(*e*) A digital call option with strike price K and maturity date T pays one unit if it finishes in the money, $S_T > K$, and nothing otherwise. Use risk-neutral valuation to show that the price of the digital call option is

$$e^{-rT} N(d_2) \qquad \text{where} \qquad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

(*f*) What is the price of the equivalent digital put?

END

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SOLUTIONS

Section A

1. (*a*) The intrinsic value is $\max[S_0 - K, 0]$. The time value is $c_0 - \max[S_0 - K, 0]$.

   (*b*) the payoff at maturity is $\max[S_T - K, 0]$.

   (*c*) If $C > S_0$ one can buy the stock and sell the call for an inflow of cash. At any date thereafter if the call is exercised the stock is sold for a price of K and if the call isn't exercised the stock is still valued. Thus at any date up to maturity there is always a positive value to the portfolio.

2. Value at Risk is the inverse of Roy's index (shortfall probability). Conditional value at risk looks at the expected value of the loss for a given probability of loss and is inversely related to the Domar-Musgrave index (expected shortfall).

3. (*a*) The Delta is 3/5 and the amount borrowed is 30.

   (*b*) The price of the call is $(3/5)150 - 30 = 60$.

4. The bull spread has a payoff of

$$\text{payoff at maturity} = \begin{cases} 0 & \text{if } S_T < K_1 < K_2 \\ S_T - K_1 & \text{if } K_1 < S_T < K_2 \\ K_2 - K_1 & \text{if } K_1 < K_2 < S_T. \end{cases}$$

The payoff at maturity is always non-negative and the portfolio must trade at a positive price, hence $C_{K_1} > C_{K_2}$. If the discounted value $e^{-rT}(K_2 - K_1)$ is borrowed in addition to the bull spread the payoff is always non-positive. Thus

$$C_{K_1} - C_{K_2} - e^{-rT}(K_2 - K_1) < 0$$

so that the difference in prices is less than the difference in the strike prices.

5. Barrier options have a payoff that depends on whether the underlying asset reaches a certain level, the barrier, prior to maturity. There are two main varieties of barrier option. The knock-in only pays out if the price of the underlying reaches the barrier and the knock-out only pays out if the underlying does *not* reach the barrier. These can be further classified by whether the barrier is set above or below the initial value of the underlying asset. If the barrier is above the initial value of the underlying, it is said to be an up option. If the barrier is below the initial value of the underlying asset, it is said to be a down option. The payoff at maturity for a down-and-out call option is

$$c_T^{down-out} = \begin{cases} c_T & \text{if } S_t > B \text{ for all } \ t \leq T \\ 0 & \text{if } S_t < B \ \text{ for any } \ t \leq T \end{cases}$$

where B is the barrier and c_T is the value of the plain vanilla call option. Clearly if you own both a down and out call option together with a down an in call option on the same underlying with the same barrier, strike prices and maturity, then you have a plain vanilla call option. So for calls and puts and down and up options:

$$\text{vanilla} = \text{in} + \text{out}.$$

6. A situation where the forward price is below the expected spot price, $F < \mathrm{E}[S_T]$, is called backwardation. A reverse situation where the forward price is above the expected spot price is called contango. Consider an investment of $F/(1+r)$ in a risk-free investment now which will deliver F at time T to offset his/her obligations on the forward contract. The cash-flow is thus a certain $F/(1+r)$ now and an uncertain amount S_T at time T. The value of the future cash flow is

$$\frac{\mathrm{E}[S_T]}{(1+r^*)}$$

where $r^* = r + \beta(\bar{r}_M - r)$ is the required rate of return, β is the *beta* of the underlying asset and $\bar{r}_M$ is the expected rate of return on the market portfolio. Then the value of the investors portfolio is

$$-\frac{F}{(1+r)} + \frac{\mathrm{E}[S_T]}{(1+r^*)}.$$

In the absence of arbitrage this portfolio has zero value. Thus we must have that the forward price satisfies

$$F = \mathrm{E}[S_T]\left(\frac{(1+r)}{(1+r^*)}\right).$$

For most assets the beta of the asset is positive and hence $r^* > r$. That is to say the asset has some systematic risk that cannot be diversified away and hence an expected return higher than the risk-free rate is required to compensate. In this typical or normal case

$$F < \mathrm{E}[S_T]$$

and we have backwardation. If the returns on the market were negatively correlated with the underlying asset then we would have $\beta < 0$ and hence $F > \mathrm{E}[S_t]$ and hence contango.

7. We have from differentiating the put-call parity condition

$$\begin{aligned}\Delta_P &= \Delta_C - 1\\ \Gamma_P &= \Gamma_C\\ \Theta_P &= \Theta_C - rKe^{-rT}.\end{aligned}$$

Section B

8. (*a*) We have on differentiation

$$\frac{\partial F}{\partial S} = s^{r(T-t)}; \quad \frac{\partial F}{\partial t} = -rS(t)e^{r(T-t)}; \quad \frac{\partial^2 F}{\partial S^2} = 0.$$

Hence substituting into Ito's lemma

$$\begin{aligned} dF &= \left(e^{r(T-t)}\mu S(t) - rS(t)e^{r(T-t)}\right) dt + \sigma S(t)e^{r(T-t)} dz \\ &= (\mu - r)F(t)dt + \sigma F(t)dz. \end{aligned}$$

(*b*) Book work — based on constructing a risk-free portfolio of short one derivative and long delta units of the underlying.

9. (*a*) The ending stock prices are $462.482, 244.743, 129.517$ and the intermediate prices are $340.03, 179.943$. The risk-neutral probability is $p = 0.501351$.

(*b*) The price of the European put at the node U is 2.51843 and at node D is 60.255. At the initial node the put price is 30.0811.

(*c*) At node D the put could immediately be exercised for a profit of 70.057. So this will be the price and hence the price of the American option at the initial node is 34.7772.

10. (*a*) Book work.

(*b*) Book work.

(*c*) The call is a convex function of the stock price.

(*d*) (i) The call price is 16.7038.

(ii) The amount borrowed is 106.373.

(iii) As the call is riskier than the underlying asset.

11. (*a*) $\Delta z = \epsilon\sqrt{\Delta T}$ where ϵ is a standard normal variable. The values of Δz are independent for any two different short time intervals.

(*b*) Then $z(T) - z(0) \sim \phi(0, \sqrt{T})$. Hence

$$\mathrm{E}[\ln S_T - \ln S_0] = \left(\mu - \frac{\sigma^2}{2}\right) T$$

and

$$\mathrm{Var}[\ln S_T - \ln S_0] = \sigma^2 \mathrm{Var}[z(T) - z(0)] = \sigma^2 T.$$

(*c*)

$$\mathrm{E}[\eta] = \frac{1}{T}\mathrm{E}\left[\ln S_T - \ln S_0\right] = \mu - \frac{\sigma^2}{2}$$

and

$$\mathrm{Var}[\eta] = \frac{1}{T^2}\mathrm{Var}\left[\ln S_T - \ln S_0\right] = \frac{\sigma^2 T}{T^2} = \frac{\sigma^2}{T}.$$

Thus the standard deviation of η is $\sigma/\sqrt{T}$.

(*d*) η is the continuously compounded rate of return an μ is the expected is the expected return.

(*e*) We have

$$\ln S_T \sim \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right).$$

The digital option pays out one unit if $\ln S_T > \ln K$. We therefore use risk-neutral valuation and replace μ by r to find the probability of payout is given by the probability that the standard normal variable is greater than

$$x = \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

This probability is given by $N(-x)$. Thus the value of the call is simply this probability discounted. That is $C = e^{-rT}N(d_2)$ as required.

(*f*) The put-call parity condition for digital options is $C + P = e^{-rT}$. So $P = e^{-rT}(1 - N(d_2)) = e^{-rT}N(-d_2)$.