KEELE UNIVERSITY

HIGHER DEGREE EXAMINATIONS, 2008

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MSc FINANCE AND MANAGEMENT
MSc FINANCE AND INFORMATION TECHNOLOGY

FIN-40008
FINANCIAL INSTRUMENTS

EXAMINATION

Candidates should attempt FOUR questions from Section A, and TWO questions from Section B. Section A is worth 60 marks and Section B is worth 40 marks.

When presenting numerical results, please give a complete step-by-step presentation of your derivations. All mathematical derivations should be accompanied by brief explanatory remarks and interpretations.

The use of hand-held, battery-operated electronic calculators will be permitted, subject to the regulations governing their use which are displayed outside the examination room. The type of calculator must be specified on the cover sheet of your answer book.

FIN 40008: FINANCIAL INSTRUMENTS

Candidates should attempt FOUR questions from Section A, and TWO questions from Section B. Section A is worth 60 marks and Section B is worth 40 marks.

SECTION A

Answer FOUR questions in this section. Each question carries 15 marks.

1. Suppose you buy an American put option at price P on an underlying asset with current price S_0. The option expires at date T and has a strike price of K.

   (a) What is the intrinsic and the time value of the option?

   (b) Describe the payoff you have at the expiration date T.

   (c) Explain why it can be expected that $P \leq K$.

2. Outline two of the following exotic options and explain the similarities and differences between them.

   Lookback options

   Asian options

   Barrier options

   "As you like it" options

3. Consider the one period binomial model with parameters $1 + u = 1.2$ and $1 + d = 0.9$ (i.e., where the rate of return on the stock is either 20% or -10%). Suppose that the stock price is initially 100 and that the risk-free interest rate is 1/8 per period (12.5%).

   (a) Consider a call option with a strike price of 100 and expiry in one period. Construct the Δ-hedge needed to create a risk-less portfolio of the stock and a written call. Verify that the payoffs to the strategy are risk-free.

   (b) Calculate the cost of the portfolio in part (a) and hence value the call option.

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4. Given the put-call parity condition $c - p = S_0 - Ke^{-rT}$ for European options, derive the put-call parity condition $S_0 - Ke^{-rT} \geq C - P \geq S_0 - K$ for American options on a non-dividend paying stock (lower case c and p denote the prices of European options and upper case C and P denote the prices of American options). Under what circumstances does $C = P$?

5. What is meant by normal backwardation and contango? Why is normal backwardation normal?

6. Explain what is meant by Delta, Theta, Gamma and Vega in option pricing.

7. There is a one-year forward contract on an asset that provides no income over the year. The current spot price is 450, the price of a forward contract is 500 and the risk-free rate of interest over the year is 10%.

   (a) Show there is an arbitrage opportunity by selling the forward contract. Carefully explain all cash flows now and at maturity that lead to an arbitrage profit.

   (b) Calculate the theoretical forward price in the absence of any arbitrage opportunities.

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SECTION B

Answer TWO questions in this section. Each question carries 20 marks.

8. An Ito process is a stochastic process of the form

$$dx = a(x,t)dt + b(x,t)dz$$

where t is time and z is a Wiener process. Ito's lemma then shows that any function $G(x,t)$ follows the process

$$dG = \left(\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right)dt + \frac{\partial G}{\partial x}b(x,t)dz.$$

It is often assumed that stock prices follow a special type of an Ito process, the geometric Brownian motion process described by

$$dS = \mu S\,dt + \sigma S\,dz$$

where S is the stock price, μ is the expected return and σ is the volatility.

(*a*) Use Ito's lemma, and the assumption of geometric Brownian motion for the stock price to show that

$$d\ln S = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\,dz.$$

(*b*) Assume that a derivative and the underlying can be traded to construct a risk-free portfolio. Hence use Ito's lemma again to derive the Black-Scholes-Merton partial differential equation

$$rc = \frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2.$$

Give a brief explanation of the formula. Explain (without giving the derivation) how this is related to the Black-Scholes formula.

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9. ABC stock has an initial price of 10. Next period it will either rise to 20 or rise to 12 with equal probability. The risk-free rate of return is 40% (2/5) and the expected return on the market portfolio is 50% (1/2). There is a call option on ABC stock which expires in one period with a strike price of $X = 16$.

   (*a*) Calculate the expected rate of return and the volatility of ABC stock. Calculate the the risk premium and *beta* of ABC stock.

   (*b*) Calculate the risk-neutral probabilities. Hence value the call option.

   (*c*) Calculate the risk-premium and volatility for the call option. Calculate the *beta* of the call option.

   (*d*) Calculate the Δ of the option. Briefly explain what Δ measures.

   (*e*) Calculate the option elasticity and explain why the option elasticity for a call option is larger than one.

10. Adopting standard notation the Black-Scholes formula for the price of a European call option is

$$c_0 = S_0 N(d_1) - e^{-rT} K N(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and $N(x)$ is the cumulative probability distribution for a standard normally distributed random variable.

   (*a*) Give a brief intuitive explanation for the formula and explain the strategy that could be used to replicate the call.

   (*b*) Use the put-call parity condition to derive the Black-Scholes formula for the equivalent put option.

   (*c*) What happens to the call price as $\sigma \to 0$?

   (*d*) What happens to the call price as $T \to 0$?

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11. Assume that the log stock price is a generalised Wiener process

$$d \ln S = \eta\, dt + \sigma dz$$

where z is a Wiener process.

(*a*) What are the two main properties of a Wiener process?

(*b*) Explain why this process is often used as a model of stock price behaviour.

(*c*) A digital call option with strike price K and maturity date T pays one unit if it finishes in the money, $S_T > K$, and nothing otherwise. Use risk-neutral valuation to show that the price of the digital call option is

$$e^{-rT} N(d_2) \qquad \text{where} \qquad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

(*d*) What is the price of the equivalent digital put?

END

FIN 40008: FINANCIAL INSTRUMENTS

SOLUTIONS

Section A

1. (*a*) The intrinsic value is $\max[K - S_0, 0]$. The time value is $P - \max[K - S_0, 0]$.

   (*b*) the payoff at maturity is $\max[K - S_T, 0]$.

   (*c*) If $P > K$ then one can sell the put and lend K for an immediate cash inflow of $P - K$. At any time that the put is exercised we cash in the loan, use the principal to pay for the stock and get an asset worth S_t and still have the interest on the loan. If the option is never exercised, then we still have the loan repayment of principal plus interest at maturity. Thus we are assured positive cash inflows in the future (even if the underlying asset ends up valueless) and thus there is an arbitrage opportunity.

2. The "as-you-like-it" option is an option where the holder can decide at a specific time whether the option is a put or a call option. Suppose that the where the decision must be made is t. The value of the option at this time is $\max[c_t, p_t]$. For European options the as-you-like-it option therefore consist of a portfolio of a call option with strike price X and maturity at date T and a put option with strike price $\frac{X}{(1+r)}$ and maturity at date t.

   Barrier options have a payoff that depends on whether the underlying asset reaches a certain level, the barrier, prior to maturity. There are two main varieties of barrier option. The **knock-in** only pays out if the price of the underlying reaches the barrier and the **knock-out** only pays out if the underlying does *not* reach the barrier. These can be further classified by whether the barrier is set above or below the initial value of the underlying asset. If the barrier is above the initial value of the underlying, it is said to be an **up** option. If the barrier is below the initial value of the underlying asset, it is said to be a **down** option. The payoff at maturity for a **down-and-out call** option is

$$c_T^{down-out} = \begin{cases} c_T & \text{if } S_t > B \text{ for all } t \leq T \\ 0 & \text{if } S_t < B \text{ for any } t \leq T. \end{cases}$$

where B is the barrier and c_T is the value of the plain vanilla call option.

Lookback options have a payoff that depends on the maximum or minimum value that the underlying asset reaches during the lifetime of the option. The payoff to maturity of a European-style lookback call is $S_T - S_{min}$ where S_{min} is the minimum value that the underlying achieves over the option's lifetime. The payoff to maturity of a European-style lookback put is $S_{max} - S_T$ where S_{max} is the maximum value that the underlying achieves over the option's lifetime. Note that if you hold both a lookback put and a lookback call your payoff is $S_{max} - S_{min}$. It is like you bought when the price was lowest and sold when the price was highest. Of course such sweet deals are likely to be costly.

An Asian option has a payoff that depends on the average price of the underlying asset from its starting date. Thus the intrinsic value of an average price Asian call option is $\max[0, S_{ave} - X]$, where S_{ave} is the average value of the asset from the start of the option to the current date. Another type of Asian option is the average strike price type, where the strike price is the average value. In this case the intrinsic value of the call is $\max[0, S_t - S_{ave}]$ and the intrinsic value of the put is $\max[S_{ave} - S_t, 0]$.

3. The call option has a payoff of 20 in the up-state and zero in the down state. Thus the payoff from a portfolio of Δ units of the underlying and one written call has a payoff of $120\Delta - 20$ in the upstate and 90Δ in the down-state. Creating a riskless portfolio means choosing Δ such that $120\Delta - 20 = 90\Delta$ or $\Delta = 2/3$. The value of this portfolio at maturity is hence 60 and with an interest rate of 1/8 its present value is 160/3. Since the value of the portfolio is $100\Delta - c$ where c is the price of the option, we have $(200/3) - c = (160/3)$ or $c = 40/3 \approx 13.33$.

Position	**Value**	$S_T < X$	$X < S_T$
Buy Call	C_t	0	$S_T - X$
Write put	$-P_t$	$S_T - X$	0
Short Stock	$-S_t$	$-S_T$	$-S_T$
Save X	X	$X(1+r)$	$X(1+r)$
Overall	$C_t - P_t - S_t + X$	rX	rX

Table 1: American Put-Call Parity: $C_t - P_t \geq S_t - X$

4. The first inequality follows from three facts. First the parity condition for European puts and calls that $c_t - p_t = S_t - (X/(1+r))$. Second the fact that American call options are not exercised early, so $C_t = c_t$ and third the fact that the American put cannot be worthless than the European put, that is $P_t \geq p_t$. Hence combining these three facts we get $c_t - p_t = S_t - (X/(1+r)) \geq C_t - P_t$. To see the second inequality consider the portfolio of a long call, a short put, shorting the stock and saving X. The payoffs to this portfolio are summarised in the Table 1. Since the portfolio yields a certain return of rX at maturity, the portfolio must have a positive value at date t, and hence we can conclude that $C_t - P_t \geq S_t - X$.

   Note the bounds are quite tight and become tighter the smallers is the interest rate r. If the interest rate is zero ($r = 0$) then the inequalities becomes an equality

$$C_t - P_t = S_t - X.$$

   If in addition the option is set up at date t and the strike price is set so that the option is at-the-money (that is $X = S_t$) then we have exact parity between the put and call prices for American options too and $C_t = P_t$.

5. A situation where the forward price is below the expected spot price, $F < \mathrm{E}[S_T]$, is called backwardation. A reverse situation where the forward price is above the expected spot price is called contango. Consider an investment of $F/(1+r)$ in a risk-free investment now which will deliver F at time T to offset his/her obligations on the forward

contract. The cash-flow is thus a certain $F/(1+r)$ now and an uncertain amount S_T at time T. The value of the future cash flow is

$$\frac{\mathrm{E}[S_T]}{(1+r^*)}$$

where $r^* = r + \beta(\bar{r}_M - r)$ is the required rate of return, β is the *beta* of the underlying asset and $\bar{r}_M$ is the expected rate of return on the market portfolio. Then the value of the investors portfolio is

$$-\frac{F}{(1+r)} + \frac{\mathrm{E}[S_T]}{(1+r^*)}.$$

In the absence of arbitrage this portfolio has zero value. Thus we must have that the forward price satisfies

$$F = \mathrm{E}[S_T]\left(\frac{(1+r)}{(1+r^*)}\right).$$

For most assets the beta of the asset is positive and hence $r^* > r$. That is to say the asset has some systematic risk that cannot be diversified away and hence an expected return higher than the risk-free rate is required to compensate. In this typical or normal case

$$F < \mathrm{E}[S_T]$$

and we have backwardation. If the returns on the market were negatively correlated with the underlying asset then we would have $\beta < 0$ and hence $F > \mathrm{E}[S_t]$ and hence contango.

6. Delta measures the change in the price of the derivative as the price of the underlying changes.
   Theta measures how the option price changes as the time to maturity approaches. It can be shown that $\Theta_C < 0$ for a non-dividend paying stock. That is the price of the option declines as maturity approaches or that longer dated options are more valuable. This is obvious for American options.
   Gamma measures how much Δ changes as the price of the underlying asset S changes and thus provides information about the appropriate dynamic hedging strategy and how Δ should be changed as S changes.

If Γ is large then it will be necessary to change Δ by a large amount as S changes. In this case it will be risky to leave Δ unchanging even over shorter periods. On the other hand if Γ is small then the costs of leaving Δ unchanged will be relatively small.
Vega measures how the option price changes as volatility changes.

7. If we sell the forward contract today, borrow 450 and use the proceeds to buy the stock we have a zero cash flow today. At maturity will deliver the stock we own and receive a price of 500 no matter what the price of the stock. This is enough to repay the principal plus the interest on the loan at 10% $(450 + 45)$ and still leave an arbitrage profit of 5. For there to be no arbitrage profit, the forward price should be 495 $(F = S_0(1 + r))$, the forward value of the current price.

Section B

8. (*a*) Let $\ln S(t) = x(t)$, so $S(t) = G(x,t) = e^x$. Then upon differentiating

$$\frac{\partial G}{\partial x} = e^x = S, \qquad \frac{\partial^2 G}{\partial S^2} = e^x = S, \qquad \frac{\partial G}{\partial t} = 0.$$

Hence using Ito's lemma

$$d\,S(t) = (\nu S(t) + 0 + \frac{1}{2}\sigma^2 S(t))dt + \sigma S(t)\,dz$$

or

$$d\,S(t) = (\nu + \frac{1}{2}\sigma^2)S(t)\,dt + \sigma S(t)\,dz.$$

(*b*) Since $c(S(t),t)$ is just a function we can apply Ito's lemma to derive

$$d\,c = \left(\frac{\partial c}{\partial t} + \mu S\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2}\right)dt + \sigma S\frac{\partial c}{\partial S}\,dz.$$

We now consider a portfolio of one option and Δ units of the underlying itself. The process for this portfolio is

$$d(c+\Delta S) = \left(\frac{\partial c}{\partial t} + \mu S\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S} + \Delta\mu S\right)dt + \left(\frac{\partial c}{\partial S} + \Delta\right)\sigma S\,dz.$$

Now setting $\Delta = -\partial c/\partial S$ eliminates the random dz term to give

$$d(c+\Delta S) = \left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S}\right)dt.$$

This is ourΔ-hedged portfolio which has eliminated all risk. Since this portfolio is riskless it must satisfy exactly the same equation as the money-market account, i.e.

$$d(c+\Delta S) = r(c+\Delta S)dt$$

and hence we have by equating these two terms that

$$\left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S}\right) = r\left(c - \frac{\partial c}{\partial S}\right)$$

where we've replace Δ by $-\partial c/\partial S$. Rewriting we have

$$rc = \left\{\left(\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S}\right)\right\}.$$

9. (*a*) The rate of return on the stock is either 100% or 20% with equal probability. Hence the expected rate of return is 60%. Thus the volatility is 40%. The risk premium is 20% and since the risk premium on the market portfolio is 10%, the *beta* of the stock is $\beta_S = 2$.

   (*b*) The risk neutral probabilities must satisfy

$$\frac{20\rho + 12(1-\rho)}{\frac{7}{5}} = 10$$

   which gives $\rho = 1/4$. The call option has a payoff of 4 in the up-state and nothing otherwise. Hence the price of the call is

$$\frac{\frac{1}{4}4}{\frac{7}{5}} = \frac{5}{7}.$$

   (*c*) The rate of return on the call is either 23/5 (460%) or -100%. Hence the expected return is 180% and the volatility is 280%. The risk-premium is 140% so the *beta* of the call is $\beta_C = 14$.

   (*d*) Delta measures the change in the call value relative to the change in the price of the underlying. Hence $\Delta = 4/8 = 1/2$.

   (*e*) The option elasticity is given by

$$\Omega = \frac{\Delta c}{\Delta S}\frac{S}{c} = \Delta\frac{S}{c} = \frac{1}{2}\frac{10}{\frac{5}{7}} = 7.$$

   The elasticity is greater than one as the call option is riskier than the underlying as is equivalent to a levered position in the underlying.

10. (*a*) The formula can be rewritten as

$$c_0 = S_0\Delta - B$$

   where Δ is the amount of the underlying asset bought and B is the amount of money borrowed needed to synthesize the call option. From the formula therefore $N(d_1)$ is the hedge parameter indicating the number of shares bought and $e^{-rT}KN(d_2)$

indicates the amount of cash borrowed to part finance the share purchase. Since $0 \leq N(d) \leq 1$ the formula shows that the replicating portfolio consists of a fraction of the underlying asset and a positive amount of cash borrowed. The Δ of this formula is also found by partially differentiating the call price formula

$$\frac{\partial c_0}{\partial S_0} = \Delta.$$

It is the slope of the curve relating the option price with the price of the underlying asset. The cost of buying Δ units of the stock and writing one call option is $S_0\Delta - c_0$. This portfolio is said to be **delta neutral** as a small change in the stock price will not affect the value of the portfolio.

(b) The Black-Scholes formula for the price of the put option at date $t = 0$ prior to maturity is given by the put-call parity condition

$$p_0 = c_0 + e^{-rT}K - S_0 = e^{-rT}K(1 - N(d_2)) - S_0(1 - N(d_1))$$

where d_1 and d_2 are defined above. By the symmetry of the standard normal distribution $N(-d) = (1 - N(d))$ so the formula for the put option is usually written as

$$p_0 = e^{-rT}KN(-d_2) - S_0N(-d_1).$$

(c) In this case the underlying asset becomes riskless so grows at the constant rate of r. Thus the future value of the stock is $S_T = e^{rT}S_0$ and the payoff to the call option at maturity is $\max[e^{rT}S_0 - K, 0]$. Thus the value of the call at date $t = 0$ is $\max[S_0 - e^{-rT}K, 0]$. To see this from the formula first consider the case where $S_0 - e^{-rT}K > 0$ or $\ln(\frac{S_0}{K}) + rT > 0$. Then as $\sigma \to 0$, d_1 and $d_2 \to +\infty$ and hence $N(d_1)$ and $N(d_2) \to 1$. Thus $c_0 \to S_0 - e^{-rT}K$. Likewise when $S_0 - e^{-rT}K < 0$ or $\ln(\frac{S_0}{K}) + rT < 0$ then d_1 and $d_2 \to -\infty$ as $\sigma \to 0$. Hence $N(d_1)$ and $N(d_2) \to 0$ and so $c_0 \to 0$. Thus combining both conditions $c_0 \to \max[e^{rT}S_0 - K, 0]$ as $\sigma \to 0$.

(d) At maturity $c_T = \max[S_T - K, 0]$ so we need to show that as $T \to 0$ the formula converges to $c_0 = \max[S_0 - K, 0]$. If $S_0 > K$

then $\ln(\frac{S_0}{K}) > 0$ so that as $T \to 0$, d_1 and $d_2 \to +\infty$. Thus $N(d_1)$ and $N(d_2) \to 1$. Since $e^{-rT} \to 1$ as $T \to 0$ we have that $c_0 \to S_0 - K$ if $S_0 > K$. Alternatively if $S_0 < K$ then $\ln(\frac{S_0}{K}) < 0$ so that as $T \to 0$, d_1 and $d_2 \to -\infty$ and hence $N(d_1)$ and $N(d_2) \to 0$. Thus $c_0 \to 0$ if $S_0 < K$. This is precisely as expected. If the option is in the money at maturity, $S_0 > K$, it is exercised for a profit of $S_0 - K$ and if it is out of the money, $S_0 < K$, the option expires unexercised and valueless.

11. (*a*) The two properties are: 1) $\Delta z = \epsilon\sqrt{\Delta T}$ where ϵ is a standard normal variable; 2) The values of Δz are independent for any two different short time intervals.

(*b*) Follows from the efficient market hypothesis and is generally consistent with empirical evidence.

(*c*) Since $z(T) - z(0) \sim \phi(0, \sqrt{T})$, we have

$$\text{E}[\ln S_T - \ln S_0] = \eta T$$

and

$$\text{Var}[\ln S_T - \ln S_0] = \sigma^2 \text{Var}[z(T) - z(0)] = \sigma^2 T.$$

Hence

$$\ln S_T \sim \phi\left(\ln S_0 + \eta T, \sigma\sqrt{T}\right).$$

By lognormality $\eta = \mu - \sigma^2/2$ where μ is the expected rate of return. Hence

$$\ln S_T \sim \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right).$$

The digital option pays out one unit if $\ln S_T > \ln K$. We therefore use risk-neutral valuation and replace μ by r to find the probability of payout is given by the probability that the standard normal variable is greater than

$$x = \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

This probability is given by $N(-x)$. Thus the value of the call is simply this probability discounted. That is $C = e^{-rT}N(d_2)$ as required.

(d) The put-call parity condition for digital options is $C + P = e^{-rT}$.
So $P = e^{-rT}(1 - N(d_2)) = e^{-rT}N(-d_2)$.