FIN 40008: FINANCIAL INSTRUMENTS

SOLUTIONS

Section A

1. (*a*) The intrinsic value is $\max[S_0 - K, 0]$. The time value is $c_0 - \max[S_0 - K, 0]$.

   (*b*) the payoff at maturity is $\max[S_T - K, 0]$.

   (*c*) If $C > S_0$ one can buy the stock and sell the call for an inflow of cash. At any date thereafter if the call is exercised the stock is sold for a price of K and if the call isn't exercised the stock is still valued. Thus at any date up to maturity there is always a positive value to the portfolio.

2. Value at Risk is the inverse of Roy's index (shortfall probability). Conditional value at risk looks at the expected value of the loss for a given probability of loss and is inversely related to the Domar-Musgrave index (expected shortfall).

3. (*a*) The Delta is 3/5 and the amount borrowed is 30.

   (*b*) The price of the call is $(3/5)150 - 30 = 60$.

4. The bull spread has a payoff of

$$\text{payoff at maturity} = \begin{cases} 0 & \text{if } S_T < K_1 < K_2 \\ S_T - K_1 & \text{if } K_1 < S_T < K_2 \\ K_2 - K_1 & \text{if } K_1 < K_2 < S_T. \end{cases}$$

The payoff at maturity is always non-negative and the portfolio must trade at a positive price, hence $C_{K_1} > C_{K_2}$. If the discounted value $e^{-rT}(K_2 - K_1)$ is borrowed in addition to the bull spread the payoff is always non-positive. Thus

$$C_{K_1} - C_{K_2} - e^{-rT}(K_2 - K_1) < 0$$

so that the difference in prices is less than the difference in the strike prices.

5. Barrier options have a payoff that depends on whether the underlying asset reaches a certain level, the barrier, prior to maturity. There are two main varieties of barrier option. The knock-in only pays out if the price of the underlying reaches the barrier and the knock-out only pays out if the underlying does *not* reach the barrier. These can be further classified by whether the barrier is set above or below the initial value of the underlying asset. If the barrier is above the initial value of the underlying, it is said to be an up option. If the barrier is below the initial value of the underlying asset, it is said to be a down option. The payoff at maturity for a down-and-out call option is

$$c_T^{down-out} = \begin{cases} c_T & \text{if } S_t > B \text{ for all } \ t \le T \\ 0 & \text{if } S_t < B \ \text{ for any } \ t \le T \end{cases}$$

where B is the barrier and c_T is the value of the plain vanilla call option. Clearly if you own both a down and out call option together with a down an in call option on the same underlying with the same barrier, strike prices and maturity, then you have a plain vanilla call option. So for calls and puts and down and up options:

$$\text{vanilla} = \text{in} + \text{out}.$$

6. A situation where the forward price is below the expected spot price, $F < \mathrm{E}[S_T]$, is called backwardation. A reverse situation where the forward price is above the expected spot price is called contango. Consider an investment of $F/(1+r)$ in a risk-free investment now which will deliver F at time T to offset his/her obligations on the forward contract. The cash-flow is thus a certain $F/(1+r)$ now and an uncertain amount S_T at time T. The value of the future cash flow is

$$\frac{\mathrm{E}[S_T]}{(1+r^*)}$$

where $r^* = r + \beta(\bar{r}_M - r)$ is the required rate of return, β is the *beta* of the underlying asset and $\bar{r}_M$ is the expected rate of return on the market portfolio. Then the value of the investors portfolio is

$$-\frac{F}{(1+r)} + \frac{\mathrm{E}[S_T]}{(1+r^*)}.$$

In the absence of arbitrage this portfolio has zero value. Thus we must have that the forward price satisfies

$$F = \mathrm{E}[S_T]\left(\frac{(1+r)}{(1+r^*)}\right).$$

For most assets the beta of the asset is positive and hence $r^* > r$. That is to say the asset has some systematic risk that cannot be diversified away and hence an expected return higher than the risk-free rate is required to compensate. In this typical or normal case

$$F < \mathrm{E}[S_T]$$

and we have backwardation. If the returns on the market were negatively correlated with the underlying asset then we would have $\beta < 0$ and hence $F > \mathrm{E}[S_t]$ and hence contango.

7. We have from differentiating the put-call parity condition

$$\begin{aligned}\Delta_P &= \Delta_C - 1\\ \Gamma_P &= \Gamma_C\\ \Theta_P &= \Theta_C - rKe^{-rT}.\end{aligned}$$

Section B

8. (*a*) We have on differentiation

$$\frac{\partial F}{\partial S} = s^{r(T-t)}; \quad \frac{\partial F}{\partial t} = -rS(t)e^{r(T-t)}; \quad \frac{\partial^2 F}{\partial S^2} = 0.$$

Hence substituting into Ito's lemma

$$\begin{aligned} dF &= \left(e^{r(T-t)}\mu S(t) - rS(t)e^{r(T-t)}\right) dt + \sigma S(t)e^{r(T-t)} dz \\ &= (\mu - r)F(t)dt + \sigma F(t)dz. \end{aligned}$$

(*b*) Book work — based on constructing a risk-free portfolio of short one derivative and long delta units of the underlying.

9. (*a*) The ending stock prices are $462.482, 244.743, 129.517$ and the intermediate prices are $340.03, 179.943$. The risk-neutral probability is $p = 0.501351$.

(*b*) The price of the European put at the node U is 2.51843 and at node D is 60.255. At the initial node the put price is 30.0811.

(*c*) At node D the put could immediately be exercised for a profit of 70.057. So this will be the price and hence the price of the American option at the initial node is 34.7772.

10. (*a*) Book work.

(*b*) Book work.

(*c*) The call is a convex function of the stock price.

(*d*) (i) The call price is 16.7038.

(ii) The amount borrowed is 106.373.

(iii) As the call is riskier than the underlying asset.

11. (*a*) $\Delta z = \epsilon\sqrt{\Delta T}$ where ϵ is a standard normal variable. The values of Δz are independent for any two different short time intervals.

(*b*) Then $z(T) - z(0) \sim \phi(0, \sqrt{T})$. Hence

$$\mathrm{E}[\ln S_T - \ln S_0] = \left(\mu - \frac{\sigma^2}{2}\right) T$$

and

$$\mathrm{Var}[\ln S_T - \ln S_0] = \sigma^2 \mathrm{Var}[z(T) - z(0)] = \sigma^2 T.$$

(*c*)

$$\mathrm{E}[\eta] = \frac{1}{T}\mathrm{E}\left[\ln S_T - \ln S_0\right] = \mu - \frac{\sigma^2}{2}$$

and

$$\mathrm{Var}[\eta] = \frac{1}{T^2}\mathrm{Var}\left[\ln S_T - \ln S_0\right] = \frac{\sigma^2 T}{T^2} = \frac{\sigma^2}{T}.$$

Thus the standard deviation of η is $\sigma/\sqrt{T}$.

(*d*) η is the continuously compounded rate of return an μ is the expected is the expected return.

(*e*) We have

$$\ln S_T \sim \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right).$$

The digital option pays out one unit if $\ln S_T > \ln K$. We therefore use risk-neutral valuation and replace μ by r to find the probability of payout is given by the probability that the standard normal variable is greater than

$$x = \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

This probability is given by $N(-x)$. Thus the value of the call is simply this probability discounted. That is $C = e^{-rT}N(d_2)$ as required.

(*f*) The put-call parity condition for digital options is $C + P = e^{-rT}$. So $P = e^{-rT}(1 - N(d_2)) = e^{-rT}N(-d_2)$.