Options

These notes consider the way put and call options and the underlying can be combined to create **hedges**, **spreads** and **combinations**. We will consider the parity condition that applies for put and call option and consider *arbitrage bounds* that apply to the prices of puts and calls.

Keywords: American option, European option, early exercise, put-call parity, hedge, spread, combination, bull spread, bear spread, strap, arbitrage bounds, covered call, protective put.

Reading: Chapters 9 and 10 from Hull.

1   Option Trading Strategies

There are basically three varieties of trading strategies. A **hedge** combines an option with an underlying stock to protect either the option or the stock against loss. A **spread** combines options of the same class but different series, where some are bought and others written. A **combination** combines options of different types, both bought or both written, on the same underlying stock.

A hedge

Suppose that you write a call option. The payoff to a call option at maturity is $\max[0, S_T - X]$ where X is the strike price and S_T is the price of the underlying at maturity. Your payoff at maturity having written the call is therefore $\min[0, X - S_T]$. If the price of the underlying rises above the strike price you will make a loss. Moreover, these losses are potentially unlimited.

If the price rises far above the strike price you are in a very exposed position. You may want to hedge or cover this risk. The obvious way to do so is to buy the stock which you will be obliged to deliver if the call is exercised. Taking a long position in the stock to hedge a written call is known as writing a **covered call**.

The payoff at maturity to writing a covered call is $\min[S_T, X]$. This is far less risky than the naked short position in the call itself as now the worst that can happen is that you sell the underlying at a price below its market value. Of course the hedged position is costly, it costs $S_t - c_t$, whereas the naked position has a positive initial inflow of cash of c_t the price of the call option.

Likewise if you buy a put option there will loss if the price of the underlying rises equal to the cost of the put option p_t. The payoff at maturity will be $\max[0, X - S_T]$. However, you will have to buy the stock if you wish to exercise the right to sell at the strike price X. Thus you may wish to buy the stock in advance at the cost of S_t. In this case the payoff at maturity is $\max[S_T, X]$ and the initial cost is $S_t + p_t$. Going long in the put and the stock is known as a **protective put** strategy.

A spread

There are many examples of spreads. **Vertical spreads** use options of the same maturity but different strike prices; **horizontal spreads** use options with the same strike prices but different maturities and diagonal spreads use option with both different strike prices and maturities.[1]

One popular type of vertical spread is the **bull spread** which is a position taken by an investor who thinks that the price of the underlying will rise.

---

[1]The terminology is derived from the way the options are presented in the financial press.

Position	$S_T < X_1 < X_2$	$X_1 < S_T < X_2$	$X_1 < X_2 < S_T$
Long Call	0	$S_T - X_1$	$S_T - X_1$
Short Call	0	0	$X_2 - S_T$
Overall	0	$S_T - X_1 > 0$	$X_2 - X_1 > 0$

Table 1: Bull Spread

A bull spread can be created by buying a call with a low strike price and writing a call with a higher strike price. Suppose that the written call has a strike price of X_2 and the long call has a strike price of $X_1 < X_2$. The payoffs at maturity are given in the Table 1. The value of the spread is $c^1 - c^2$, where c^1 is the price of the call option with a strike price of X_1 and c^2 is the price of the call option with a strike price of X_2. Since the bull spread always yields a non-negative payoff at maturity (it is a limited liability asset) it follows that $c^1 > c^2$.

Another type of spread is a **bear spread**. A bear spread can be created by holding a put with a higher strike price and writing a put with a lower strike price. In this case the investor benefits if the markets is falling. This is in contrast to a bull spread where the investor gains when the market rises.

Exercise: An investor buys a March put on Reed Elsevier with a strike price of 649 for a price of 55.5 and writes a March put on Reed Elsevier with a strike price of 609 for a price of 34. Calculate the payoff now and the possible payoffs in March when the puts expire. Draw a table and a diagram to illustrate the payoffs.

A combination

A combination involves taking a position in both puts and calls on the same underlying asset. One particular combination is a **strap**. A strap consists of a long position in two calls and one put with the same strike price and

Position	Value	$S_T < X$	$X < S_T$
Long Call	c_t	0	$S_T - X$
Short Forward	$-v_t$	$X - S_T$	$X - S_T$
Overall	$c_t - v_t$	$X - S_T$	0

Table 2: Put-Call Parity

expiration date. This position is appropriate if it is thought that there is likely to be a large change in the price of the underlying but the direction of change is unknown. However it is thought that an increase is more probable than a decrease. The payoff at maturity from the strap is $X - S_T$ if $S_T < X$ and $2(S_T - X)$ if $S_T > X$.

2   Put-Call Parity Condition (European)

Consider buying a call option with a strike price of X and a maturity date of T and at the same time going short in a forward contract on the same underlying and with a delivery price of X.

The overall position from this portfolio is the same as a long position in an equivalent put option on the same underlying with the same strike price and maturity date. Thus the two portfolios must have equal values, $c_t - v_t = p_t$ where v_t is the value of the forward contract. From our previous knowledge of forward contracts we know that $v_t = S_t - (X/(1+r))$. Therefore

$$(1) \qquad\qquad c_t + \frac{X}{(1+r)} = p_t + S_t$$

where r is shorthand for r_t^T. This condition is the put-call parity condition from European options. This equation is known as the put-call parity condition as it was first derived under the assumption that the options were at the money, $S_t = X$ and that the interest rate was 0.

It can be seen directly from this equation that the portfolio which is long in the stock and long in the put option is equivalent to a portfolio which is long in the call and long in a discount bond with a face value of X.

Exercise: Check it out.

Another way to see this is as follows. Consider simultaneously buying a call and writing a put. If $S_T > X$ then exercise the call for a profit of $S_T - X$. If $X > S_T$, then the put will be exercised resulting in a payoff of $X - S_T$ for the holder and a payoff of $S_T - X$ for the writer. Thus the portfolio of one call and one written put results in a payoff of $S_T - X$ in all circumstances. The value of this payoff can be evaluated using the stochastic discount factor k. The value of the portfolio at time t is

$$c_t - p_t = E[k \cdot (S_T - X)] = E[k \cdot S_T] - X E[k] = S_t - \frac{X}{(1+r)}$$

since by definition the appropriately discounted value of the stock price at time T is the stock price now, S_t, and $E[k]$ is the appropriate discount factor. Again we have the same put-call parity condition.

To labour the point once more remember that the payoff at maturity from writing a covered call is $\min[S_T, X]$. Suppose that in addition you go short in a risk free bond with a face value of X. The overall payoff at maturity is $\min[S_T - X, 0]$, the same as writing a put option. The value of this portfolio is $S_t - c_t - \frac{X}{(1+r)}$, which must equal the value of writing the put option, $-p_t$.

3 Arbitrage Bounds

We will know consider the payoff to the option prior to maturity. We will establish bounds for the option value prior to maturity.

The **intrinsic value** or **parity value** of an option at time t is the payoff to the option if the current date were the maturity date. Thus the intrinsic

value of the call option at time t is $\max[0, S_t - X]$ where s_t is the current price of the underlying asset and the intrinsic value of a put option is $\max[X - S_t, 0]$.

An American option will always trade at a price at or above its intrinsic value, since with an American option it is always possible to exercise the option now and realise the intrinsic value. The difference between the price of an option an its intrinsic value is known as the **premium** or **time value** of the option. Thus the price of an option is the sum of its intrinsic value plus its premium

$$\text{price of option} = \text{intrinsic value} + \text{time value}.$$

If an option is initially set up at the money, $S_0 = X$ then the intrinsic value of the option is 0 and the premium and price are equivalent. This accounts for why the option price is sometimes referred to as the premium. We will use the term time value to avoid confusion.

A European call option also has a positive time value. That is the price of a European call option cannot be less than its intrinsic value. Remember that the payoff to the call option is $Max[S_T - X, 0]$. The call can be valued using the stochastic discount factors k. Thus the value of the call option at time t is $c_t = E[k \cdot Max[S_T - X, 0]]$. Clearly

$$c_t \geq E[k \cdot (S_T - X)] = E[k \cdot S_T] - XE[k] = S_t - \frac{X}{(1+r)}$$

where the first inequality comes from the definition of the maximum and the other equalities follow as $E[k \cdot S_T] = S_t]$, the stochastically discounted value of the stock at maturity is its current value, and the fact that $E[k] = \frac{1}{(1+r)}$. Equally since $S_T \geq 0$ and $X \geq 0$, it follows that $Max[S_T - X, 0] \leq S_T$, so $c_t \leq E[k \cdot S_T] = S_t$. Since the call cannot command a negative price we have for the arbitrage bounds for a call option

$$(2) \qquad\qquad S_t \geq c_t \geq \max\left[S_t - \frac{X}{(1+r)}, 0\right].$$

Position	Value	$S_T < X$	$X < S_T$
Write Call	$-c_t$	0	$X - S_T$
Buy Stock	S_t	S_T	S_T
Overall	$S_t - c_t$	S_T	X

Table 3: Call Upper Bound: $S_t \geq c_t$

Since the strike price on the RHS is discounted we have $\frac{X}{(1+r)} < X$, so that

$$c_t \geq \max\left[S_t - \frac{X}{(1+r)}, 0\right] > \max[S_t - X, 0]$$

for $t < T$ and $r > 0$, so the European call has value greater than its intrinsic value at any date prior to maturity.

To see how the inequalities in (2) can be derived without using the stochastic discount factor first let's consider the upper bound on the European call option. The value of the call cannot be greater than the value of the stock. This is obvious because the call option only gives you the right to buy the stock. If the call option cost more than the stock, it would simply be cheaper to buy the stock now and have nothing further to pay. To see this more clearly consider writing a covered call, that is buy the stock and write a call option. The payoff now is $c_t - S_t$, which is positive if the call price is greater than the current stock price. At maturity, either the call goes unexercised, if $S_T < X$, and we are left with the stock with value of S_T, or the call is exercised, in which case we give up the stock we own for the agreed strike price of X. Either way we end up with a positive payoff at maturity, so the value of the portfolio must be positive $S_t - c_t > 0$ if $X > 0$ or $S_T > 0$. If $c_t > S_t$ there is an arbitrage possibility formed by writing a covered call. The payoffs are summarised in the Table 3.

To see the lower bound on the call option, $c_t \geq S_t - \frac{X}{(1+r)}$ recall that we have already shown that a long position in the call option with a strike price of X combined with a short position in a forward contract on the same

Position	Value	$S_T < X$	$X < S_T$
Buy Call	c_t	0	$S_T - X$
Sell Stock	$-S_t$	$-S_T$	$-S_T$
Buy Bond	$\frac{X}{(1+r)}$	X	X
Overall	$c_t - S_t + \frac{X}{(1+r)}$	$X - S_T$	0

Table 4: Call Lower Bound: $c_t \geq S_t - \frac{X}{(1+r)}$

underlying and with a delivery price of X is equivalent to a long position in a put option with the same strike price. A short position in the forward contract with a delivery price of X is itself equivalent to selling the stock short now and buying a risk-less bond with a face value of X. The payoffs are summarised in Table 4. The portfolio has a payoff of 0 if $X < S_T$: the bond pays out X and the call is exercised using the bond payoff to pay the strike price and the purchase of the stock closes out the short position. The portfolio has a payoff of $X - S_T$ if $S_T < X$ as the call option goes unexercised. Thus the portfolio replicates the put option and in particular never has a negative payoff. The value of the portfolio must therefore be positive by the no-arbitrage principle, $c_t \geq S_t - \frac{X}{(1+r)}$.

An American option cannot be less valuable than the equivalent European option as the American option provides all the same rights as the European option *and* the opportunity to exercise early. Thus $C_t \geq c_t$ and $P_t \geq p_t$. However, although it is possible to exercise an American call option on a non-dividend paying stock prior to maturity, it is never optimal to do so. This is because the European call option has a positive time value and thus its price is greater than its intrinsic value. Since $C_t \geq c_t > S_t - X$ for $t < T$, it will always be better to sell the call option and get C_t rather than exercise it and get $S_t - X$. Thus the option to exercise the American option early is redundant and the American and European options on non-divided paying assets are equivalent and hence must trade at the same price, $C_t = c_t$.

The qualification that the underlying asset should be non-dividend paying *is* important. It may be optimal to exercise an American option early if the underlying asset is a dividend paying stock. Let D_t denote the present value of dividends paid on the underlying during the remainder of the options life. Then the lower bound for the call option is $S_t - D_t - \frac{X}{(1+r)}$. If D_t is large enough, then this lower bound can below the intrinsic value of $S_t - X$. In this case it may be optimal to exercise early. To take an extreme case suppose that the call is in the money but the firm on which the option is written is unexpectedly to be liquidated and all the asset value is to be paid out in dividends. Then after the dividend payment, the stock is worthless and the call option can never be in the money. In this case it is best to exercise early and take the positive value of exercise now.

By analogy the arbitrage bounds for a European put option on a non-dividend paying stock with a payoff of $Max[X - S_T, 0]$ are

$$X \geq p_t \geq \max \left[\frac{X}{(1+r)} - S_t, 0 \right].$$

Note that an American put option cannot be less than its intrinsic value, so the arbitrage bounds for an American put option are

$$X \geq P_t \geq \max[X - S_t, 0].$$

For an American put option early exercise can be optimal even if the underlying asset pays no dividends. To take an extreme example, suppose that $S_t = 0$ but $X > 0$. It is impossible to gain more by waiting since the stock price cannot fall further. In addition the gain to be had by waiting is discounted at the risk-free rate, and if the interest rate is positive it is better to exercise now rather than wait. If exercise is ever optimal before maturity then $P_t = X - S_t$ since if $P_t > X - S_t$ it would be better to sell the option rather than exercise it.

Exercise: Find an arbitrage opportunity if $p_t > X$.

Exercise: Find an arbitrage opportunity if $p_t < \frac{X}{(1+r)} - S_t$.

Position	Value	$S_T < X$	$X < S_T$
Buy Call	C_t	0	$S_T - X$
Write put	$-P_t$	$S_T - X$	0
Short Stock	$-S_t$	$-S_T$	$-S_T$
Save X	X	$X(1+r)$	$X(1+r)$
Overall	$C_t - P_t - S_t + X$	rX	rX

Table 5: American Put-Call Parity: $C_t - P_t \geq S_t - X$

4   Put-Call Parity Condition (American)

There is a put-call parity condition for American options on non-dividend paying stocks. The condition however, is one of inequalities rather than equality and is given by:

$$(3) \qquad\qquad S_t - \frac{X}{(1+r)} \geq C_t - P_t \geq S_t - X.$$

The first inequality follows from three facts. First the parity condition for European puts and calls (equation (1)) that $c_t - p_t = S_t - (X/(1+r))$. Second the fact that American call options are not exercised early, so $C_t = c_t$ and third the fact that the American put cannot be worthless than the European put, that is $P_t \geq p_t$. Hence combining these three facts we get $c_t - p_t = S_t - (X/(1+r)) \geq C_t - P_t$. To see the second inequality consider the portfolio of a long call, a short put, shorting the stock and saving X. The payoffs to this portfolio are summarised in the Table 5. Since the portfolio yields a certain return of rX at maturity, the portfolio must have a positive value at date t, and hence we can conclude that $C_t - P_t \geq S_t - X$.

Note the bounds in equation (3) are quite tight and become tighter the smallers is the interest rate r. If the interest rate is zero ($r = 0$) then the inequalities in equation (3) becomes an equality

$$C_t - P_t = S_t - X.$$

If in addition the option is set up at date t and the strike price is set so that the option is at-the-money (that is $X = S_t$) then we have exact parity between the put and call prices for American options too and $C_t = P_t$.