

Forward and Futures Contracts

These notes explore forward and futures contracts, what they are and how they are used. We will learn how to price forward contracts by using *arbitrage* and *replication* arguments that are fundamental to derivative pricing. We shall also learn about the similarities and differences between forward and futures markets and the differences between forward and futures markets and prices. We shall also consider how forward and future prices are related to spot market prices.

Keywords: *Arbitrage, Replication, Hedging, Synthetic, Speculator, Forward Value, Maintainable Margin, Limit Order, Market Order, Stop Order, Backwardation, Contango, Underlying, Derivative.*

Reading: You should read Hull chapters 1 (which covers option payoffs as well) and chapters 2 and 5.

1 Background

From the 1970s financial markets became riskier with larger swings in interest rates and equity and commodity prices. In response to this increase in risk, financial institutions looked for new ways to reduce the risks they faced. The way found was the development of exchange traded derivative securities. Derivative securities are assets linked to the payments on some underlying security or index of securities. Many derivative securities had been traded over the counter for a long time but it was from this time that volume of trading activity in derivatives grew most rapidly.

The most important types of derivatives are futures, options and swaps. An option gives the holder the right to buy or sell the underlying asset at a specified date for a pre-specified price. A future gives the holder the

obligation to buy or sell the underlying asset at a specified date for a pre-specified price. Swaps allow investors to exchange cash flows and can be regarded as a portfolio of futures contracts.

Options and futures are written on a range of major stocks, stock market indices, major currencies, government bonds and interest rates. Most options and futures in the UK are traded on the London International Financial Futures Exchange (<http://www.liffe.com/>) and most options and futures in the US are traded on the Chicago Board of Trade (<http://www.cbot.com/>). It is also possible to trade futures contracts on a range of different individual stocks from across the world at Euronext (<http://www.universal-stockfutures.com/>). Such exchange traded derivatives might be described as off-the-peg or generic as the details of the derivative are specified by the Exchange House. Other derivatives are traded over the counter. They are tailor-made and designed specifically to meet the needs of individual traders, the party and counter party.

Derivatives allow investors a great deal of flexibility and choice in determining their cash flows and thus are ideal instruments for hedging of existing risk or speculating on the price movements of the underlying asset. Thus for example it is possible to offset the risk that a stock will fall in price by buying a put option on the stock. It is possible to gain if a large change in the price of the underlying asset is anticipated even if the direction of change is unknown. It is also possible by using an appropriate portfolio of options to guarantee that you buy at the lowest price and sell at the highest price — a trader's dream made reality.

2 Forward Contracts

Forwards and futures contracts are a special type of derivative contract. Forward contracts were initially developed in agricultural markets. For example an orange grower faces considerable price risk because they do not know at what price their crops will sell. This may be a consequence of weather condi-

tions (frost) that will affect aggregate supply. The farmer can insure or hedge against this price risk by selling the crop forward on the forward orange concentrate market. This obligates the grower to deliver a specific quantity of orange concentrate at a specific date for a specified price. The delivery and the payment occur only at the forward date and no money changes hands initially. Farmers can, in this way, eliminate the price risk and be sure of the price they will get for their crop. An investor might also engage in such a forward contract. For an example an investor might sell orange concentrate forward for delivery in March at 120. If the price turns out to be 100, the investor buys at 100 and delivers at 120 making a profit of 20. If the weather was bad and the price in March is 150, the investor must buy at 150 to fulfill her obligation to supply at 120, making a loss of 30 on each unit sold. The farmer is said to be a **hedger** as selling the orange concentrate forward reduces the farmer's risk. The investor on the other hand is taking a position in anticipation of his beliefs about the weather and is said to be a **speculator**. This terminology is standard but can be misleading. The farmer who does not hedge their price risk is really taking a speculative position and it is difficult to make a hard and fast distinction between the two types of traders.

Why trade forward?

For an investor the forward market has both pros and cons. The advantage is that there is no initial investment. That is it costs nothing now to buy or sell forward. The disadvantage is that there is a change of suffering a large loss.

The price of a forward contract

Let's consider a forward contract for a particular underlying asset, e.g. IBM stock, with a maturity date of T . The price of such a forward contract is easy to determine. In the absence of any transactions or storage cost the price of the forward contract is the **future value** of the current spot price.

	Position	Cost Now	Payoff at Maturity
(1)	Long Underlying	S_0	S_T
(2)	Long Forward	0	$(S_T - F)$
(3)	Short Forward	0	$(F - S_T)$
(4)	Long Discount Bond	$\frac{F}{(1+r)}$	F

Table 1: Forward Price

Let F be the forward price and S_0 be the current spot price and let r_0^T be the risk-free rate of interest from now until the maturity date T , then

$$F = (1 + r_0^T)S_0.$$

To simplify the notation denote r_0^T simply by r . Then $F = (1 + r)S_0$. To see that this formula is correct, let's consider the payoff and cost of the positions that can be taken on the stock, the forward contract and a risk-free discount bond with a face value of F and a maturity date of T . If we take a long position in the stock, the cost now is S_0 and the payoff at the maturity date is S_T . We don't know what S_T will be since the payoff to the asset is uncertain. If we take a long position in the forward contract, the cost now is zero and the payoff at maturity is $S_T - F$. That is we are obligated to buy at F and the underlying asset can be sold for S_T . If $S_T > F$, then we buy at F and can immediately sell at S_T for a profit of $S_T - F$. If on the other hand $S_T < F$ then we must buy at F but can only sell at S_T so we have a loss since $S_T - F < 0$. In addition suppose we buy the discount bond. It costs $F/(1 + r)$ now and pays out the face value F at maturity. We can summarise all this information in Table 1.

Now consider the following trading strategy: go long in the forward contract and buy the discount bond. The payoff to the forward contract is $(S_T - F)$, you are committed to buy at F but can sell for S_T , and the payoff to the bond is simply F , so the total payoff is S_T . Thus this trading strategy *replicates* the payoff to the underlying asset. We then invoke the *arbitrage* principle that since this trading strategy has the same payoff at maturity as the underlying asset, it must cost exactly the same as buying the underlying asset itself. The strategy costs $F/(1 + r)$ as the forward contract involves no

Position	Cost Now	Payoff at Maturity
(1) + (3) = (4)	S_0	F
(2) + (4) = (1)	$\frac{F}{(1+r)}$	S_T

Table 2: Forward

initial outlay, therefore it must be the case that $F/(1+r) = S_0$ or

$$F = (1+r)S_0.$$

That is, the forward price is simply the future value of the stock. The long position in the stock (1) is equivalent to a portfolio of a long position in the forward and a long position in the discount bond (2) + (4).

As an alternative suppose you go long in the stock and short on the forward contract, that is a portfolio of (1) and (3). The overall payoff at maturity is $S_T + (F - S_T) = F$. The cost of this strategy is S_0 but has the same payoff as a risk-free bond with face value of F and is thus equivalent to position (4). Thus as before $F/(1+r) = S_0$. This is summarised by Table 2.

As yet another possibility consider buying the underlying stock and going short on the discount bond with face value of F (that is borrow an amount $F/(1+r)$). At maturity one has an asset worth S_T but an obligation to repay F and thus a net worth of $S_T - F$. This is exactly the same as the long forward contract. Since the payoffs are the same we are said to have *synthesized* or replicated the forward contract. The cost of this *synthetic* forward contract is the cost of the stock now S_0 less what we borrowed, $F/(1+r)$, so that the net cost is

$$S_0 - \frac{F}{(1+r)}.$$

The payoff is the same as the forward contract. Yet the forward contract involves no exchange of money upfront. So the cost of the synthetic forward must be zero too:

$$S_0 - \frac{F}{(1+r)} = 0$$

which again delivers the same conclusion that $F = (1+r)S_0$.

Exercises:

Exercise 1: If $F > (1+r)S_0$ identify an arbitrage opportunity. Put together a portfolio which gives you money now and only offsetting obligations at the maturity date.

Exercise 2: If $F < (1+r)S_0$ identify an arbitrage opportunity. Put together a portfolio which gives you money now and only offsetting obligations at the maturity date.

Exercise 3: If the stock pays out a dividend D at the maturity date T , so the total payoff to holding the stock is $S_T + D$, calculate the forward price if the interest rate is r

Forward Value

The forward contract is initially negotiated so that there is no initial outlay. That is the delivery price on the forward contract is chosen so that the value of the contract is zero. However, as maturity approaches the price of the underlying asset changes but the delivery price does not. Thus as time progresses the forward contract may have a positive or negative value. Let K be the delivery price and let S_t denote the price of the underlying asset at time t with time $T - t$ left to maturity. The forward price is $F_t = S_t(1 + r_t^T)$ where r_t^T is the risk-free interest rate from t until T .

The same argument can be used above can now be used to find how the value of the forward contract changes as the time moves to maturity. Let this value be v_t . Consider the portfolio of one long forward contract and the purchase of a discount bond with face value of K and maturity date of T . The payoff to the forward contract is $(S_T - K)$ but the payoff to the bond is K leaving a net payoff of S_T . The cost of this portfolio is $v_t + \frac{K}{(1+r_t^T)}$. Since it replicates exactly the underlying (which has a price of S_t) we have

$$v_t = S_t - \frac{K}{(1 + r_t^T)} = \frac{(F_t - K)}{(1 + r_t^T)}$$

where the last part follows since $S_t = F_t/(1 + r_t^T)$. To check that this makes sense first consider what happens at $t = 0$. At $t = 0$ the delivery price is chosen so $v_t = 0$, that is $K = S_0(1 + r_0^T) = F_0$ and the forward price is equal to the delivery price. Next consider $t = T$. Then $r_t^T = 0$ and we get $v_T = S_T - K$ which is just the payoff to the forward contract at maturity.

To presage what we will do subsequent, the value of the forward contract can also be calculated by using the **stochastic discount factor** k .¹ A forward contract with a delivery price of K has a payoff at maturity of $S_T - K$. Thus the value of this payoff is

$$v_t = E[k \cdot (S_T - K)] = E[k \cdot S_T] - KE[k] = S_t - \frac{K}{(1 + r_t^T)}$$

where the last part of the equation follows since $E[k]$ measure the appropriately discounted payoff of one unit of payoff for sure and thus is equal to the discount factor $1/(1 + r_t^T)$.

3 Futures Contracts

So far we have used the terms forward and futures interchangeably and they are equivalent if there is no interest rate uncertainty. There are however, some differences between forward and futures contracts.

Forward contracts are normally traded over the counter and futures contracts are generally exchange traded with futures prices reported in the financial press. With a futures contract therefore the exchange provides a standardised contract with a range of specified delivery periods. Thus a wheat futures contract will be specify the delivery of so many bushels of wheat for delivery in a particular month. The quality and delivery place will also be specified. The exact day of delivery within the month is usually left to the discretion of the writer of the contract.

¹We will study risk-neutral probabilities and the stochastic discount factor later in the course.

The key difference between forward and futures contracts is that forwards are settled at maturity, whereas futures contracts are settled daily. This daily settlement is done by requiring the investor to hold a *margin account* with the exchange. Thus although the contract costs nothing initially, the investor is required to deposit a certain amount of funds, the *initial margin* with the exchange. This margin account is *marked to market* to reflect the daily gains or losses on the contract. Thus for example if you buy a futures contract on Wednesday for 250 and the following day the futures price has fallen to 240, you will have suffered a loss of 10 and this amount will be deducted from your margin account. In effect the futures contract is closed out and rewritten every day. The exchange will also specify a *maintenance margin* which is the amount which must be maintained in the margin account, usually about 75% of the initial margin. If the margin account does fall below the maintenance margin the investor will be required to deposit extra funds, the *variation margin*, with the exchange. Most futures contracts are closed out prior to maturity and don't actually result in delivery of the underlying. Thus an investor will settle the futures contract and withdraw the amount in the margin account on that day.

Traders on futures (and other types of exchange markets) can place conditional trade as well as trade orders. There are three main types of order that can be executed. (i) A "Market" order will trade *immediately* at the current market price once the order is made there is no turning back! (ii) "Limit" orders are used to set a price at which the trader is prepared to trade. For example if the prices are currently high, the trader can input a price a bit lower than the current offer, and place a conditional order to buy. The order will now move into a *working orders account* and will be executed *if* the offer price falls to the limit level specified. (iii) "Stop" orders enable the trader to limit losses in his/her portfolio. This involves setting a conditional price at which to sell the asset if the market moves too much in the wrong direction. The trader specifies a price and volume at which to sell. The order will again be placed in the working orders account and will be executed if the price falls to the level specified.

Day	Position	Futures Price	Gain/loss	Future Value
0	Long $\frac{1}{(1+r)^{T-1}}$	G_0	-	-
1	Long $\frac{1}{(1+r)^{T-2}}$	G_1	$\frac{G_1 - G_0}{(1+r)^{T-1}}$	$G_1 - G_0$
2	Long $\frac{1}{(1+r)^{T-3}}$	G_2	$\frac{G_2 - G_1}{(1+r)^{T-2}}$	$G_2 - G_1$
...
$T-2$	Long $\frac{1}{(1+r)}$	G_{T-2}	$\frac{G_{T-2} - G_{T-3}}{(1+r)^2}$	$G_{T-2} - G_{T-3}$
$T-1$	Long 1	G_{T-1}	$\frac{G_{T-1} - G_{T-2}}{(1+r)}$	$G_{T-1} - G_{T-2}$
T	0	G_T	$G_T - G_{T-1}$	$G_T - G_{T-1}$

Table 3: Equivalence of Futures and Forward Prices

Although futures are settled daily we now show that the forward price and the futures price are the same if interest rates are known even if the forward contract cannot be traded. For simplicity suppose that the interest rate is constant and let r denote the daily interest rate and suppose that the futures contract matures on date T and let $G_0, G_1, G_2, \dots, G_{T-1}, G_T$ denote the futures price on each of the trading days. Suppose that we have a position of $\frac{1}{(1+r)^{T-t-1}}$ on day t . Thus initially our position is $\frac{1}{(1+r)^{T-1}}$ and we increment our position on a daily basis. The change in the value of our position on day 1 is

$$\frac{G_1 - G_0}{(1+r)^{T-1}}.$$

To calculate the future value of this gain or loss compounded to date T we multiply by $(1+r)^{T-1}$ as there are $T-1$ days to go. Thus the future value is simply $G_1 - G_0$. The positions and future value on each day is summarised in Table 3.

Overall the future value from this portfolio is $G_T - G_0$. Since the futures price at maturity must equal the spot price, $G_T = S_T$, the future value is $S_T - G_0$. If we combine this portfolio with a the purchase of a risk-free asset at time zero with a face value of G_0 the portfolio has a payoff of S_T . Since the futures contracts cost nothing to purchase the overall cost of the portfolio is the cost of the risk-free asset, $\frac{1}{(1+r)^T}$. Suppose that the forward contract has a delivery price of F_0 . Since we have already seen that the forward contract combined with a risk-free asset with face value of F_0 gives at time

T a portfolio worth S_T , these two portfolios must cost the same. Hence

$$\frac{G_0}{(1+r)^T} = \frac{F_0}{(1+r)^T}$$

or $G_0 = F_0$ and hence for any t , $G_t = F_t$. Thus the forward and futures prices are equivalent.

This argument can be replicated if the interest rate changes in a known way, simply by choosing the appropriate positions so that the future value of the gain or loss on the futures contract is $G_t - G_{t-1}$ on date t .

There is however, a difference between futures and forward prices if the interest rate is uncertain. Suppose that there is a positive covariance between the interest rate and the price of the underlying asset. Then if the price of the asset rises the gains on the futures contract will tend to be valued at a high interest rate and similarly losses on the futures contract will be valued at a low interest rate. An investor holding a forward contract is not affected by changes in the interest rate if they cannot trade the forward. Hence if the covariance between the interest rate and the underlying is positive, the futures price will tend to be higher than the forward price. In practice even this difference is likely to be small for as most futures contracts are held for relatively short durations. Thus for most practical purposes there is little difference between the forward and the futures price.

It is to be remembered too that although we've talked about the market price there are really two prices, the *bid* price and the *offer* price.² The offer price is the price one can buy at (the market offers the contract at this price) and the bid price is the price one can sell at (how much the market is prepared to pay for the asset). It must be the case that **offer price** \geq **bid price** otherwise you could buy at the offer price and sell at the bid price and make an immediate arbitrage profit. The difference between the bid and offer prices is known as the **bid-offer spread** and the cost of buying at the offer price as selling at the bid price is known as the **roundtrip** cost.

²The offer price is also known as the *ask* price, particularly in the U.S.

Backwardation and Contango

We have shown that the forward price is just the future value of the underlying, $F = (1 + r_0^T)S_0$. Perhaps surprisingly the expected value of the underlying asset at time T , $E[S_T]$ does not affect the forward price at all. Yet it is also clear that the forward price tends to the current spot price as the contracts tends to maturity. To see this suppose that we are very close to the delivery period. If the forward price were below the current spot price, then it would be possible buy the forward contract, wait for delivery and sell almost surely at the higher spot price. This will tend to drive the forward price up toward the spot price. Similarly if the forward price were above the spot price one could sell the forward contract, buy at the lower spot price and make the delivery generating an almost sure profit. With everyone doing this the forward price will fall toward the spot price.

A situation where the forward price is below the expected spot price, $F < E[S_T]$, is called **backwardation**. A reverse situation where the forward price is above the expected spot price is called **contango**. An interesting question, and one that exercised Hicks and Keynes in the 1930s, is whether forward prices normally exhibit backwardation or contango and why.

To consider the relationship between the forward price and the expected future spot price it is necessary to consider the risk involved in holding a forward or futures position. Consider an investor or speculator who holds a long position in a forward contract. This obligates the investor to pay out F the forward price at maturity in return for an asset that will be worth the unknown amount S_T . For simplicity suppose the investor $F/(1 + r)$ in a risk-free investment now which will deliver F at time T to offset his/her obligations. The cash-flow is thus a certain $F/(1 + r)$ now and an uncertain amount S_T at time T .

The question that remains is how to value the risky future payoff S_T . As in corporate finance we could evaluate this payoff using a CAPM so that the

value of the future cash flow is

$$\frac{E[S_T]}{(1+r^*)}$$

where $r^* = r + \beta(\bar{r}_M - r)$ is the required rate of return, β is the *beta* of the underlying asset and \bar{r}_M is the expected rate of return on the market portfolio. Then the value of the investors portfolio is

$$-\frac{F}{(1+r)} + \frac{E[S_T]}{(1+r^*)}.$$

As the asset is priced so that $S_0 = E[S_T]/(1+r^*)$, if this term were positive or negative, there would be an arbitrage opportunity. For example if $F/(1+r) > E[S_T]/(1+r^*)$ then there would be an arbitrage opportunity to borrow $F/(1+r)$, buy the underlying asset at the price $S_0 = E[S_T]/(1+r^*)$ and short the forward contract. This would create a net inflow of cash today and offsetting cash flows at the maturity date as illustrated in Table 4. Thus we must have that the forward price satisfies

$$F = E[S_T] \left(\frac{(1+r)}{(1+r^*)} \right).$$

If the *beta* of the underlying asset were zero then $r^* = r$ and the forward price would be equal to the expected spot price. In these circumstances we would say the forward price is an **unbiased predictor** of the expected future spot price. We do however, know that for most assets the beta of the asset is positive and hence $r^* > r$. That is to say the asset has some systematic risk that cannot be diversified away and hence an expected return higher than the risk-free rate is required to compensate. In this typical case

$$F < E[S_T]$$

and we have backwardation.³ If the returns on the market were negatively correlated with the underlying asset then we would have $\beta < 0$ and hence $F > E[S_t]$ and hence contango.

The same argument can also be made by using the stochastic discount factor and hence does not rely on a specific pricing model such as the CAPM.

³Since it is the typical case it is often referred to as *normal backwardation*.

Position	Cost Now	Payoff at Maturity
Long Underlying	$S_0 = E[S_T]/(1 + r^*)$	S_T
Short Forward	0	$(F - S_T)$
Short Discount Bond	$-\frac{F}{(1+r)}$	$-F$

Table 4: Arbitrage Possibility

Since the current stock value reflects the appropriately stochastically discounted value of possible future values, $S_0 = E[k \cdot S_T]$. Since $F = S_0(1 + r)$ we have that $F = (1 + r)E[k \cdot S_T]$. Using the covariance rule

$$\begin{aligned} F &= (1 + r)E[k \cdot S_T] = (1 + r)E[k]E[S_T] + (1 + r)Cov(k, S_T) \\ &= E[S_T] + (1 + r)Cov(k, S_T) \end{aligned}$$

since $E[k] = 1/(1 + r)$. It can be seen from the above equation that whether there is backwardation or contango depends on the sign of $cov(k, S_T)$. Typically, because individuals are risk averse, the demand will be for assets that offer insurance and the price of returns in low payoff states will be high. Thus for most assets the covariance will be negative. Typically then $F < E[S_T]$ and the forward will exhibit backwardation.

Exercises:

Exercise 4: The current gold price is \$500 per ounce. The forward price for delivery in one year is \$575 per ounce. The cost of storing an ounce of gold for one year is \$40 and this must be paid now in advance. The risk-free rate of interest is 10% per annum. If you own ten ounces of gold, how can you exploit an arbitrage opportunity to make \$190?

Exercise 5: Consider a forward contract written on a non-dividend paying asset. The current spot price is \$65. The maturity of the contract is in 90 days and the interest rate over this period is 1.1%. Determine the forward price. What is the value of this contract? A corporate client wants a 90-day

forward contract with the delivery price set at \$60. What is the value of this contract? (See Hull p.108).

Exercise 6: Consider a one year futures contract on an underlying commodity that pays no income. It costs \$5 per unit to store the commodity with payment being made at the end of the period. The current price of the commodity is \$200 and the annual interest rate is 6%. Find the arbitrage-free price of the futures contract. (See Hull p.116).