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UNIVERSITY OF CAMBRIDGE

ECONOMIC THEORY DISCUSSION PAPER

No. 74

SELF-ENFORCING WAGE CONTRACTS

by

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and

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April 1984

SSRC Research project on Risk, Information and Quantity Signals in Economics

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SELF-ENFORCING WAGE CONTRACTS

By Jonathan Thomas and Tim Worrall[1]

1. Introduction

In recent years considerable attention has been devoted to models of implicit contracts. Both the case in which the employer and employee have symmetric information[2] and the case in which they have asymmetric information[3] have been thoroughly examined. Progress too has been made in constructing general equilibrium models with labour contracts[4] and in analysing their efficiency. One of the major defects of the implicit contract literature is the often tacit assumption that the contract is enforced by an independent third agent. It is assumed that it is infinitely costly for any agent to renege upon the contract. But if the third party can enforce the contract it is not clear why this or some other agent cannot also provide the appropriate insurance to the employer and the employee thereby making the initial contract unnecessary. The assumption that reneging is infinitely costly is clearly extreme. It is the purpose of this paper to analyse the other extreme in which there is no enforcement mechanism, so the cost of reneging is zero. Contracts negotiated under this assumption are truly implicit, they must be designed to be self-enforcing, so neither agent has an incentive to renege.

Contracts between employers and employees can be negotiated over a wide variety of parameters. For example a contract may depend upon retail prices, money supply, profits, input prices, output prices, working conditions, effort, etc.. Some of these parameters may be jointly observed by the employer, employee and a third party, e.g., retail prices. If this is true a contract dependent upon, say, retail prices can be enforced by the third party if the appropriate legal machinery exists. Even so there are likely to be large costs associated with legal enforcement. The costs of legal enforcement of labour market contracts are likely to be especially high, since the employee's principal resource is labour skill which cannot be sequestrated and employers are protected by limited liability. Contracts may also depend upon parameters which are observed by the employer and employee but which are costly for a third agent to verify, e.g. fluctuations in working conditions might fall into this category. Then even if there exists a cheap mechanism for the legal enforcement of contracts, such contracts cannot be costlessly administered. This paper is concerned with non-binding contracts which depend upon parameters observed by both employer and employee. What will not be considered are self-enforcing contracts which depend upon parameters about which the employer and employee have asymmetric information, e.g. effort.

Although this paper is confined to a discussion of self-enforcing labour contracts the model could be equally applied to other fields. Examples include the international borrowing/lending market and the market for the supply of firm specific inputs. Telser [13] gives numerous examples of idiosyncratic exchange to which the present analysis is relevant. Self-enforcing contracts are important to any exchange situation where there exists an opportunity for one or more of the parties to make a short term gain by reneging or terminating the relationship. A self-enforcing contract will have the feature that future gains derived from holding to the contract exceed the short term gain to be obtained by reneging.

In the labour market context Holmstrom [10] has analysed a finite time horizon model in which workers may quit at any time but firms are

legally bound to honour their contracts. Holmstrom shows that there does exist a contract which dominates the spot market. This contract exhibits "front-loading," that is to say, since the firm must offer the worker at least the spot market wage in the final period it recoups this loss by lower wage payments in the previous periods. Bester [4] examines stationary wage contracts in an infinite horizon model and like Holmstrom he assumes the contract can be made legally binding upon the employer but not the employee. In a slightly different context Grout [7] analyses a Nash bargaining solution in the absence of binding contracts. In his model only the workers have an incentive to renege once shareholders have commited themselves to a given level of investment. Contracts are self-enforcing in the sense that the investment level chosen depends upon the workers opportunity to renege.

The present paper is an outgrowth of the work of Bester and Holmstrom. It examines the simplest model in which the employer's and employee's self-enforcement constraints can be explicitly recognized. It is assumed that each risk neutral firm can employ one risk averse employee who provides one unit of labour inelastically. The firm converts this into one revenue unit of output. A contract specifies the wage to be paid to the employee in each contingency. The self-enforcement problem arises because there is a random spot market wage. It is assumed that the employee can renege at any time on the contract and immediately find employment at the spot market wage. Similarly the employer is always free to dispense with the contract worker and hire instead from the spot market. Once an agent buys or sells on the spot market it is assumed they are unable to return to the contract market. In fact such an arrangement is a perfect equilibrium. Suppose the worker believes that once he has reneged on the contract the employer (and all other employers) will never pay him more than the spot wage. Then it is optimal for the worker, once he has reneged, never to accept less than the spot wage. Likewise suppose the employer believes that once the employee has reneged, he will never accept less than the spot wage. Then the employer will never offer more than the spot wage. Thus the employee is no longer regarded as "trustworthy". If the same loss of confidence occurs when the firm reneges, then not reneging is a perfect equilibrium. Put this way, self-enforcing contracts can be viewed as equilibria of a repeated game with the spot market wage as the stage-Nash equilibrium. This viewpoint will not be stressed in this paper, rather the risk sharing incentives of the employer and employee will be emphasized. For example if the spot market wage is low the employer will wish to offer the employee insurance by means of a higher contract wage and also reduce the average wage bill; but at the same time it creates an incentive for the employer to renege on the contract. Therefore a self-enforcing contract must compensate the employer in this situation by offering him higher expected gains in the future.

There are a number of questions which naturally arise. Does there exist a self-enforcing contract which dominates the spot market? Is there an optimal contract? If so, is it unique? Is it efficient (will it be as good as a contract which is legally enforceable)? Will the contract be time dependent (as it is in Holmstrom's front-loading solution) or will it be history dependent (so wages at any date-event pair depend on the entire history up to that date)? If either of the latter questions are answered in the affirmative, can the long run properties of the contract be characterized? It will be shown below that the optimal contract has a particularly simple solution. This means not only is it possible to answer the questions above but to

actually compute the solution for any given set of parameters. To illustrate this a numerical example is presented.

The outline of the paper is:- Section 2 presents and discusses the model. Section 3 contains the results. The promised numerical example is given in Section 4 and Section 5 contains some concluding comments.

2. The Model

The purpose of this section is to outline the simplest model in which the self-enforcement problem arises. Some discussion of the assumptions is contained in remarks.

There is an infinite sequence of dates, $t = 0, 1, 2, \ldots \infty$ and a finite set of states, $s_t = 0, 1, 2, \ldots, S$, at each date. There are two types of agents, firms and workers. A firm can employ only one worker at each date. A worker supplies one indivisible unit of labour which the firm converts into one revenue unit of output. There are no costs or delay to a worker of moving from one firm to another and no cost to the firm of hiring or firing workers. Each agent can trade labour on the spot market or negotiate a contract at date zero which specifies the wage payment at each date-event pair. The only allowable contracts are between one firm and one worker. Each agent has perfect foresight so they know the spot market wages at each date-event pair. Every agent is a perfect competitor and so treats the spot market wage parametrically. Contracts are not legally enforceable, so each agent can renege on the contract if it would be to their advantage. The only feasible contracts are self-enforcing contracts in which neither agent has an incentive to renege at any date-event pair.

All workers are infinitely lived and have an identical per period state independent utility function, $u = u(\omega)$, where ω is the wage received. It is assumed

ASSUMPTION A1: $u = u(\omega)$: $[a, b] \to R$, $u(a)$, $u(b)$ are finite, $u(\cdot)$ is differentiable, strictly increasing and strictly concave.

These assumptions are standard. The employee is strictly risk averse so there is always the opportunity for sharing risk. All firms are identical, infinitely lived, and derive utility from profits according to the function, $v = v(\omega)$ where $v(\cdot)$ is strictly decreasing. In addition

ASSUMPTION A2: $v(\omega) = -\omega$

so firms are risk neutral. Notice the constant revenue of the firm has been incorporated into the functional form of v. Each agent discounts future utility by the common factor α.

REMARK 1: The assumption that firms are risk neutral can be easily relaxed. So too can the assumption that output is state independent. However one or other of these assumptions is needed if the solution method adopted below is to be applied. Risk neutrality is convenient in allowing comparisons with legally enforceable contracts which specify a constant wage. The common discount factor is not strictly necessary: an appropriately modified solution applies if firms and workers discount at different rates. If for example the employee discounts the future more heavily a legally enforceable contract would call for a wage which falls over time. The results below can all be interpreted relative to

this trend.

REMARK 2: It may be thought unduly restrictive to consider only contracts which are negotiated at date zero and in which both parties never renege. This is not true. It has been argued in the Introduction that self-enforcing contracts constitute a perfect equilibrium. Another way to think of this is:- a contract with reneging in some states can always be replaced by a contract which specifies the action taken by the reneging agent as the agreed action. The new contract will be self-enforcing; it will specify that the contract wage is equal to the spot market wage in some histories. The assumption that the only allowable contracts are between one firm and one worker is not restrictive. If a firm offers a contract to more than one worker another firm can always offer a better contract to one of these workers. Similarly competition among workers will ensure that no worker has a contract with more than one firm.

REMARK 3: The assumption that the cost of legal enforcement is infinitely high is an extreme assumption made for the purpose of analysis. It is conjectured that the results of the model would not be dissimilar if the costs were high but finite. Equally borrowing and lending, durable goods, divisible labour could all be introduced at the cost of considerably complicating the analysis. Some insight into these problems can be gained by examining the simplest possible case.

REMARK 4: An infinite sequence of dates is strictly necessary for there to be a self-enforcing contract. If there was a finite time horizon, T, then either the firm or worker would have an incentive to renege at T unless the contract wage were equal to the spot market wage in all possible states. Since there is no gain to be made from the contract at date T it must also be that contract wages equal spot market wages in all possible states at date $T - 1$, otherwise one or other party would have an incentive to renege. Proceeding backwards from time $T - 1$ the only self-enforcing contract pays the spot market wage at each date-event pair. The idea of infinitely lived agents is somewhat disquieting; if it is preferred $(1 - \delta)$ can be thought of as the probability that an agent dies in some period. Then the expected life of an agent is finite, although there is always a possibility that they may live for any given amount of time.

Each state is identified solely by the spot market wage $w(s)$. As a convention $w(s) > w(s - 1)$. The probability that state s occurs at date t is $p(s)$. It will be maintained:

ASSUMPTION A3: States are independently and identically distributed, with $a < w(0) < w(s) < 1 \leq b$.

The latter part of Assumption A3 ensures that utility is defined for all possible spot market wages and firms have an incentive to hire labour in every state.

REMARK 5: Assumption A3 can be weakened to allow spot market wages to follow any stationary Markov process. All that is needed for the solution methodology adopted below is that in any given state the future taking bygones as bygones looks the same at every date.

REMARK 6: Nothing so far has been said about what generates the volatility of spot market wages. There are a number of possible interpretations.

For example there may be some agents who are unable to negotiate contracts, perhaps because they are finitely lived, or because they have revealed themselves unreliable in the past. These agents are forced to trade labour on the spot market. Randomness in their preferences or technology is sufficient to generate a random spot market wage. Equally suppose there are more infinitely lived workers than firms. Then given the institutional assumption that each firm employs only one worker there will be some unemployment. Unemployed workers are forced to produce on their own. This is less efficient and more risky than labouring for a firm. The spot market wage can then be interpreted as a random opportunity cost.

Let s_t denote the state which occured at date t. A history up to time t is a list of those states which have occured up to and including time t, $h_t = (s_0, s_1, s_2, \ldots, s_t)$. Let $h_{\tau/t} = (s_t, s_{t+1}, \ldots, s_\tau)$ be the history from time t to time τ. The expectation conditional upon h_t is denoted $E[\cdot|h_t]$.

A contract δ_0 between an employer and an employee at date zero specifies a set of contract wages $(\omega(h_\tau))_{\tau=0}^{\infty}$, where $\omega(h_\tau)$ is the wage to be paid to the employee at time τ if the history is h_τ. It is helpful to consider the hypothetical contract δ_t^s made at time t, when $s_t = s$. This contract specifies a set of wages $(\omega(h_{\tau/t}))_{\tau=t}^{\infty}$, where $\omega(h_{\tau/t})$ is the wage to be paid at date τ if the history from time t is $h_{\tau/t}$. So for any given h_t, a δ_t^s (where $s_t = s$) can be derived directly from δ_0.

DEFINITION 1: A contract δ_t^s is h_t feasible if

$$(\omega(h_{\tau/t}))_{\tau=t}^{\infty} \in \Lambda_{h_t} = \Big\{ (\omega(h_{\tau/t}))_{\tau=t}^{\infty} \mid$$

$$w(s_v) - \omega(h_{v/t}) + E\Big[\sum_{\tau=v+1}^{\infty} \alpha^{\tau-v} (w(s_\tau) - \omega(h_{\tau/t}) \mid h_t\Big] \geq 0,$$

$$u(\omega(h_{v/t})) - u(w(s_v)) + E\Big[\sum_{\tau=v+1}^{\infty} \alpha^{\tau-v} (u(\omega(h_{\tau/t})) - u(w(s_\tau))) \mid h_t\Big] \geq 0,$$

$$\forall\ v \geq t, \ \forall\ h_{v/t}\Big\}.$$

A contract is h_t feasible if it is self-enforcing from date t onwards given the history h_t. There are two sets of self-enforcing constraints in Definition 1. The first set says that at any future possible date-event pair the net gain to the firm from holding to the contract is non-negative. In other words the firm has no incentive to renege even if at some $h_{v/t}$, $\omega(h_{v/t}) > w(s_v)$. The second set of constraints require that the employee has no incentive to renege either.

DEFINITION 2: A contract δ^* is optimal if it solves

$$\max_{\delta^*_0 \in \Lambda_{s_0}} E\Big[\textstyle\sum_{\tau=1}^{\infty} \alpha^\tau (- \omega(h_\tau)) \mid s_0\Big] \quad \forall \quad s_0 = 0, 1, \ldots S$$

A contract is optimal if it is feasible in each state at date zero and also maximizes the discounted sum of expected net profits.

REMARK 7: Contract wages at date t have been written as a function of the entire history up to that date, while spot market wages are independently and identically distributed. It might be thought that optimal contract wages will be memoryless and time-invariant so $\omega(h_\tau) = \omega(s)$. Bester [4], a priori, restricts contracts to be of this form. In a two period model however, Holmstrom [10] shows that optimal wages will exhibit 'front-loading'. It will be shown that optimal contract wages are history dependent. That this is so even when spot market wages are independently and identically distributed makes the result that much stronger.

REMARK 8: Definition 2 says that the optimal contract maximizes expected discounted profits subject to the feasibility constraints. Therefore the net gain to the employee from the contract is driven to zero. It is thus implicit that there is a large number of workers competing for a limited number of contracts. This is not restrictive, the model could easily be generalized to allow each firm to employ more than one worker. Then competition between firms and workers would simply allocate the net gain to be made from concluding the contract. However the assumption that the employee's net gain is zero makes many of the results more stark without affecting any of the general conclusions and is therefore included for pedagogical purposes.

3. Results

This section is divided into four parts. In Section 3.1 efficiency and optimality are examined. It is shown that the optimal self-enforcing contract is never efficient. In Section 3.2 the dynamic programming solution of the model is developed. Section 3.3 examines the structure of wages in the optimal contract. In Section 3.4 some comparative static analysis is considered.

3.1 Efficiency and Optimality

In this section it is shown that the optimal self-enforcing contract is not efficient. That is to say a legally enforceable contract could always offer each agent a larger expected utility. This result could easily be derived as a corollary to the solution given in Section 3.3. By deriving it directly it is hoped that this will provide further intuition about the nature of the optimal contract.

It is easy to see that for a discount factor sufficiently near unity, the set of feasible self-enforcing contracts will include contracts with a constant wage, ω.

PROPOSITION 1: For some $\omega \in \{\omega | \omega < \Sigma_{s=0}^{S} p(s)\, w(s),\ u(\omega) < \Sigma_{s=0}^{S} p(s) u(w(s))\}$, there exists an $\alpha^* < 1$ such that for $\alpha > \alpha^*$ the contract $(\omega(h_\tau))_{\tau=0}^{\infty} = (\omega)_{\tau=0}^{\infty} \in \Lambda$.

PROOF: Consider the gain to the firm of a constant wage contract starting from state s_t at date t.

$$w(s_t) - \omega + \sum_{\tau=1}^{\infty} \alpha^\tau \sum_{s=0}^{S} p(s)\ (w(s) - \omega) = w(s_t) - \omega$$

$$+ \frac{\alpha}{1-\alpha} \sum_{s=0}^{S} p(s)\ (w(s) - \omega)$$

Since the last term is positive by assumption, for any value of $w(s_t) - \omega$ it is possible to find some $\alpha_f^s < 1$ such that the total is non-negative for $\alpha \geq \alpha_f^s$. Likewise for the employee there exists and $\alpha_w^s < 1$ such that for $\alpha \geq \alpha_w^s$ the employee will not renege in state s. Let $\alpha^* = \max\ (\alpha_f^0 \ldots \alpha_f^s, \alpha_w^0 \ldots \alpha_w^s)$.

DEFINITION 3: A contract δ is efficient if there is no other contract which gives greater expected utility to either the employer or employee and no less to the other.

No mention of feasibility is made in Definition 3. Efficient contracts are simple to characterize.

LEMMA 1: An efficient contract specifies a constant wage $\omega(h_t) = \omega$ at each date $t = 0, 1, \ldots \infty$.

PROOF: Suppose $\omega(h_t) \neq \omega(h_\tau)$ for some h_t, h_τ. Denote prob $(h_t) = p_t$ and prob$(h_\tau) = p_\tau$. Then define a wage ω by

$$\omega = (p_t\ \ w(h_t) + \alpha^{\tau-t}\ p_\tau\ \ w(h_\tau))/(p_t + \alpha^{\tau-t}\ p_\tau)$$

By the strict concavity of $u(\cdot)$

$$(p_t + \alpha^{\tau-t}\ p_\tau)\ u(\omega) > p_t\ \ u(\omega(h_t)) + \alpha^{\tau-t}\ \ u(\omega(h_\tau))$$

so replacing $\omega(h_t)$ and $w(h_\tau)$ by ω increases the employee's utility. Similarly since the firm is risk neutral expected profits are unchanged. Therefore $\omega(h_t) \neq \omega(h_\tau)$ cannot be efficient.

Optimal self-enforcing contracts never specify a constant wage and hence cannot be efficient. They are dominated by contracts that are legally enforceable. To see this consider

PROPOSITION 2: An optimal contract gives the employee a zero net gain at date zero.

PROOF: Suppose to the contrary that the optimal contract has $u(\omega(s_0)) - u(w(s_0)) + E\left[\sum_{\tau=1}^{\infty} \alpha^{\tau} \left(u(\omega(h_\tau)) - u(w(s_\tau))\right) \middle| s_0\right] \quad \varepsilon > 0$. for some s_0. Then if $\omega(s_0) > a$, reducing it by a small amount does not violate feasibility but increases profits contrary to the assertion. If $\omega(s_0) = a$, $u(\omega(s_0)) - u(w(s_0)) < 0$. Then $E\left[\sum_{\tau=1}^{\infty} \alpha^{\tau-1} \left(u(\omega(h_\tau)) - u(w(s_\tau))\right) \middle| s_0\right] > \varepsilon/\alpha$. So for some s, the workers net gain from s onwards is strictly greater than ε/α. Again if $\omega(s_0, s_1) < a$ it can be reduced, increasing profits without violating feasibility. Otherwise look at time 2; and so on. Clearly there must exist some $\omega(h_t) > a$ s.t. $E\left[\sum_{\tau=t}^{\infty} \alpha^{\tau-t} \left(u(\omega(h_{\tau/t})) - u(w(s_\tau))\right) \middle| h_t\right] > 0$ otherwise the workers gain will become unbounded along some history which is impossible since the single period gain is bounded. Such a wage can be reduced by an appropriately small amount.

COROLLARY 1: An optimal self-enforcing wage contract is not a constant wage contract.

This is an obvious consequence of Proposition 2. If a constant wage contract was optimal it must satisfy

$$u(\omega) - u(w(s_0)) + (\alpha/(1-\alpha)) \sum_{s=0}^{\theta} p(s)(u(\omega) - u(w(s))) = 0 \quad \forall\ s_0$$

This is clearly impossible if w is random. So in the optimal contract the worker, at date zero, is just indifferent between being employed on the contract or having to trade in the spot market forever.

3.2 The Dynamic Programming Solution

This section will show how the optimal self-enforcing contract can be examined by using a dynamic programming approach. This is done by

exploiting the forward looking nature of the self-enforcing constraints. First for any history h_t it is possible to define the h_t utility possibility frontier.

DEFINITION 4: For any history h_t, $f_{h_t}(U_{h_t}): I_{h_t} \to J_{h_t}$, is the h_t utility possibility frontier if

$$f_{h_t}(U_{h_t}) = \sup \left\{ w(s_t) - \omega(h_t) + E\left[\sum_{\tau=t+1}^{\infty} \alpha^{\tau-t} \left(w(s_\tau) - \omega(h_{\tau/t})\right)\right.\right.$$

$$\phi_{h_t} \in \Lambda_{h_t}$$

$$\left.\left| h_t\right] \;\middle|\; u(\omega(h_t)) - u(s_t) + E\left[\sum_{\tau=t+1}^{\infty} \alpha^{\tau} \left(u(\omega(h_{\tau/t})) - u(w(s_\tau))\right) \middle| h_t\right] \geq U_{h_t} \right\}.$$

In Definition 4 U_{h_t} is the net expected gain to the employee derived from the contract at time t if the history is h_t. Then $f_{h_t}(U_{h_t})$ is the largest net expected gain to employer from the contract, given the history h_t and that the employee receives U_{h_t}. The set I_{h_t} is the set of U_{h_t}'s for which the constraint set is non-empty. Similarly J_{h_t} is the set of $f_{h_t}(U_{h_t})$ for which the constraint is non-empty given history h_t.

What Definition 4 tries to convey is the idea that for any history h_t the optimal contract will always correspond to some point on an appropriate utility possibility frontier. The next three lemmas are necessary to justify the use of the dynamic programming approach. Theorem 1, which follows the first of these lemmas, proves that there exists a unique optimal contract.

LEMMA 2: (i) Λ_{h_t} is independent of h_{t-1} and t.

(ii) Λ_{h_t} is a convex set $\forall\ h_t$.

(iii) f_{h_t} is independent of h_{t-1} and t.

PROOF: (i) Since all the self enforcing constraints are forward looking and since in each state the future looks the same at any date the result follows directly from Definition 1.

(ii) Consider any two contracts $(\omega(h_{\tau/t}))_{\tau=t}^{\infty}$ and $(\omega'(h_{\tau/t}))_{\tau=t}^{\infty}$ which are h_t feasible. Let $\omega^{\beta}(h_{v/t}) = \beta\, \omega(h_{v/t}) + (1 - \beta)\omega'(h_{v/t})$, $\beta \in (0, 1)$. Then at any date v, given any history h_v

$$u(\omega^{\beta}(h_{v/t})) - u(w(s_v)) \geq \beta\left[u(\omega(h_{v/t})) - u(w(s_v))\right]$$

$$+ (1 - \beta)\left[u(\omega'(h_{v/t})) - u(w(s_v))\right]$$

by the concavity of $u(\cdot)$. So the contract $(\omega^{\beta}(h_{\tau/t}))_{\tau=t}^{\infty}$ satisfies all the employees non-negatively constraints and given the risk neutrality of employers is therefore feasible.

(iii) Follows directly from (i).

Since Λ_{h_t} and f_{h_t} are independent of h_{t-1} and t, they can be notationally simplified to Λ_s and f_s respectively. Likewise I_{h_t} and J_{h_t} can be written I_s and J_s.

THEOREM 1: For each state s, and any $U_s \in I_s$, there exists a unique feasible contract which gives U_s to the employee and $f_s(U_s)$ to the employer.

PROOF: For $U_s \in I_s$ consider a sequence of feasible contracts, δ^{v}, offering the employee at least U_s and each offering a gain to the

employer of V^v such that $\lim V^v = f_s(U_s)$. A contract may be thought of as a sequence of wages, one for each history, where each wage belongs to $[a, b]$. Thus the space of all contracts is compact (and sequentially compact) in the product topology by Tychynoff's theorem. Hence there exists a subsequence of contracts $\delta^{v'}$ such that $\omega^{v'}(h_t) \to \omega^{\infty}(h_t)$ for all h_t (i.e. pointwise convergence). It must now be shown that this limiting contract δ^{∞} is feasible, gives the employee at least U_s and gives the employer $f_s(U_s)$. With strict discounting the dominated convergence theorem together with the continuity of u implies

$$\lim_{v\to\infty}\left(u(\omega^{v}(h_t)) - u(w(s_t)) + E\left[\sum_{\tau=t+1}^{\infty} \alpha^{\tau-t}(u(\omega^{v}(h_{\tau/t})) - u(w(s_{\tau})))\middle|h_t\right]\right)$$

$$= u(\lim \omega^{v}(h_t)) - u(w(s_t)) + E\left[\sum_{\tau=t+1}^{\infty} \alpha^{\tau-t}(u(\lim \omega^{v}(h_{\tau/t})) - u(w(s_{\tau})))\middle|h_t\right]$$

$$= u(\omega^{\infty}(h_t)) - u(w(s_t)) + E\left[\sum_{\tau=t+1}^{\infty} \alpha^{\tau-t}(u(\omega^{\infty}(h_{\tau/t})) - u(w(s_{\tau})))\middle|h_t\right]$$

for any h_t. Similarly for the firm's gains. Since each gain at each history on $\delta^{v'}$ is non-negative, then it is so on δ^{∞} as well. Likewise δ^{∞} gives the employee U_s. Finally since by assumption $\lim V^v = f_s(U_s)$ the above equality again gives $V^{\infty} = f_s(U_s)$. Uniqueness follows immediately from the convexity of the Λ_s and strict concavity of u.

LEMMA 3: For each state, s, I_s and J_s are compact intervals containing 0.

PROOF: It will be shown that I_s is compact; symmetric arguments show J_s to be compact. $0 \in I_s$ since paying the spot wage is always feasible. If $U \in I_s$, then $U' \in I_s$ for $U' < U$ since the constraint set is no smaller, so I_s is an interval. To prove it is closed consider a sequence $U^v \in I_s$ such that $\lim U^v = U^\infty$. There will be a corresponding sequence of contracts δ^v in each constraint set. The same argument as in Theorem 1 shows there to be a feasible limiting contract δ^∞ offering the worker at least U^∞.

Nothing is lost by restricting the intervals to non-negative numbers, so $I_s = [0, \bar{U}_s]$, $J_s = [0, \bar{V}_s]$.

LEMMA 4: Each f_s is decreasing and strictly concave on I_s

PROOF: That f_s is decreasing follows directly from Definition 4. Strict concavity follows straight-forwardly from convexity of the Δ_s and the fact that for different $U_s \in I_s$ the optimal contracts differ by Proposition 2.

The solution to the model can now be analysed using a dynamic programming approach. Bellman's principle of optimality states that if a program is optimal from date t onwards then it is also optimal from date $t + 1$ onwards. In the present context if a contract is optimal at date t state s then it will be optimal at all possible states at date $t + 1$. By "optimal" is here meant on the efficiency frontier. This principle is encapsulated in the fundamental functional equation

$$(1)\quad f_s(U_s^t) = \max_{\substack{w(s)\in[a,\, b] \\ \{U_q^{t+1}\}_{q=0}^S}} \{w(s) - \omega(s) + \alpha \sum_{q=0}^{S} p(q)\, f_q(U_q^{t+1})\}$$

$$U_q^{t+1} \geq 0, \quad f_q(U_q^{t+1}) \geq 0 \qquad q = 0, 1, \ldots S,$$

$$u(\omega(s)) - u(w(s)) + \alpha \sum_{q=0}^{S} p(q)\, U_q^{t+1} = U_s^t)$$

This equation holds for each state $s = 0, 1, \ldots S$ and each date $t = 0, 1, \ldots \infty$. The functional equation represents a strictly concave program. By assumption wages belong to a compact set $[a, b]$, and by Lemma 4 each U_q^{t+1} is chosen in the compact interval I_q. Lemma 3 shows $f_q(\cdot)$ is concave so the objective function is concave, and by the strict concavity of $u(\cdot)$ the constraint set is strictly convex. The optimal contract is derived from (1) by setting $U_s^0 = 0$ for each s and solving forward period by period.

3.3 The Optimal Wage Structure

This is the main section of the paper; its purpose is to solve for the optimal wage structure. Both the evolution of wages through time and the distribution of wages across states will be examined. The latter will be seen to be a consequence of the former and therefore it is the temporal structure of wages which will be emphasized. The model presented in Section 2 stressed the dependence of wages at date t upon the entire history of spot market wages up to that date. In this section it will be shown that the history of wages takes a particularly simple form. To be precise wages follow a stationary, finite state Markov process, which makes wages today depend upon the spot market wage today and the contract wage yesterday. Further it will be shown that contract wages are more inflexible than spot market wages.

The procedure adopted in this section is as follows. First it will be shown that the value function, $f_s(\cdot)$ of equation (1) is a differentiable function and hence the first order conditions for this functional equation are described. Second it will be shown that for each state contract wages belong to a non-empty, time-invariant interval. Third the relationship between these $S + 1$ intervals is examined. This leads to Theorem 2 which provides a simple rule to describe the evolution of wages over time. A corollary to this theorem examines the distribution of wages across states at any particular date.

Lemma 3 showed that $f_s(\cdot)$ is a concave function. A concave function is differentiable almost everywhere, the next lemma shows that $f_s(\cdot)$ is differentiable everywhere:

LEMMA 5: For each $s = 0, 1, \ldots, S$, $f_s(\cdot)$ is continuously differentiable on the open interval $(0, \bar{U}_s)$.

PROOF: For any $U_s \in (0, \bar{U}_s)$, let δ_s be a contract which gives the worker U_s and the firm $f_s(U_s)$ in state s. Consider forming a new contract δ_γ from δ_s by replacing $\omega(s_0)$ by $\omega(s_0) + \gamma$, where $\gamma \in N_\varepsilon(0)$, the ε-neighbourhood of 0. Provided $\delta(s_0) \in (a, b)$, δ_γ is certainly feasible for ε small enough. For each γ in such a $N_\varepsilon(0)$ there will be a gain to the employee and an associated gain to the employer. Let $F\colon A \to J_s$ be the function representing this association, where $A \subset I_s$ is an open interval containing U_s. F will have the form $F(U) = K - u^{-1}(U)$ where K is some constant. This function is concave, differentiable on an open neighbourhood of U_s, $F(U_s) = f_s(U_s)$, and $F(U) \leq f_s(U)$ by definition of f_s, so it satisfies the conditions of Lemma 1 of [3] and therefore f_s is differentiable at U_s. Since f_s is monotonic, it must also be continuously differentiable. If $\omega(s_0) = a$ or b then proceed as in Proposition 2

The optimal contract is completely described by the functional equation (1) and the specification of U_s^0. Since equation (1) represents a concave program the first order conditions are necessary and sufficient for a

solution. Letting $\{\alpha\, p(q)\, \phi_q^{t+1}\}_{q=0}^S$, $\{\alpha\, p(q)\, \psi_q^{t+1}\}_{q=0}^S$, and λ_s^t be the multipliers for the respective constraints in the program the first order conditions are

$$(2)\quad -\lambda_s^t = f_q'(U_q^{t+1})(1 + \psi_q^{t+1}) + \phi_q^{t+1}, \qquad q = 0, 1, \ldots S,$$

$$(3)\quad -\lambda_s^t = -1/u'(\omega(s_t)),$$

and in addition

$$(4)\quad -\lambda_s^t = f_s'(U_s^t).$$

It should again be emphasized that these equations hold for each state, $s = 0, 1, \ldots S$, and for each time period, $t = 0, 1, \ldots \infty$. The next two lemmas allow Theorem 2 to be proved which gives the rule whereby optimal wages are updated.

Lemma 6 shows that for each state, $s = 0, 1, \ldots S$, there is a time independent interval to which optimal contract wages must belong. Each interval specifies a wage $\bar{\omega}_s$ which is the maximum it is ever optimal to pay in state s, and a wage $\underline{\omega}_s$ which is the minimum.

LEMMA 6: For any history h_t, with $s_t = s$, the optimal contract wage $\omega(h_t)$ is contained in the closed, non-empty interval, $[\underline{\omega}_s, \bar{\omega}_s]$.

PROOF: Using equations (3) and (4), $\omega(h_t) = g_s(U_s^t)$, where for each s, $g_t(\cdot)$ is differentiable on $(0, \bar{U}_s)$ and strictly increasing. Monotonicity follows from the strict concavity of $u(\cdot)$ and $f_s(\cdot)$, differentiability follow by the continuous differentiability of $u(\cdot)$ and

$f_s(\cdot)$. Since $U_s^t \in [0, \bar{U}]$ from Lemma 4 wages range over the compact interval $[\underline{\omega}_s, \bar{\omega}_s]$, where $\bar{\omega}_s = g_s(\bar{U}_s)$ and $\underline{\omega}_s = g_s(0)$. Since $\bar{U}_s$ is non-negative the interval is non-empty.

The size of each interval $[\underline{\omega}_s, \bar{\omega}_s]$ will depend on all the parameters of the model; the discount factor, the parameters of the utility function and the parameters of the distribution of spot market wages. In Section 3.4 a comparative static exercise will be carried out by varying the discount factor. Lemma 7 examines, for any particular set of parameters, how each of the $S + 1$ intervals relate to each other and the spot market wage.

LEMMA 7: For any date t and any two states k and q such that $w(k) > w(q)$, then

(i) $\bar{\omega}_k > \bar{\omega}_q$, $\underline{\omega}_k > \underline{\omega}_q$, $\forall$ $q, k = 0, 1, \ldots, S$, $k > q$.

(ii) $w(s) \in [\underline{\omega}_s, \bar{\omega}_s]$ $\forall$ $s = 0, 1, \ldots S$,

and $w(0) = \underline{\omega}_0$, $w(S) = \bar{\omega}_S$

PROOF: (i) Using the functional equation (1), it follows that for any date t, any two states k and q, and any $U_q \in [0, \bar{U}_q]$

(5) $f_k(U_q + u(w(q)) - u(w(k))) = f_q(U_q) + w(k) - w(q)$,

and therefore differentiating with respect to U_q

(6) $f_k'(U_q + u(w(q)) - u(w(k))) = f_q'(U_q)$.

a. To show $\bar{\omega}_k > \bar{\omega}_q$, let $U_q = \bar{U}_q$ and $U_k = \bar{U}_k$, then using equation (5), $f_k(\bar{U}_q + u(w(q)) - u(w(k))) > 0 = f_k(\bar{U}_k)$. By the concavity

of $f_k(\cdot)$ and using equation (6), $f_q'(\bar{U}_q) > f_k'(\bar{U}_k)$. But $\omega(s_t) = k(f_s'(U_s^t))$ where $k(\cdot)$ is continuous and strictly decreasing from equations (3) and (4). Therefore $\bar{\omega}_k = k(f_k'(\bar{U}_k)) > k(f_q'(\bar{U}_q)) = \bar{\omega}_q$.

b. To show $\underline{\omega}_k > \underline{\omega}_q$, suppose $U_q = U_k = 0$, then again from (5) and (6) and concavity, $f_q'(0) > f_k'(0)$, and hence $\underline{\omega}_k = k(f_k'(0)) > k(f_q'(0)) = \underline{\omega}_q$.

(ii) a. It will first be shown that $\bar{\omega}_s \geq w(s)$ and $\bar{\omega}_S = w(S)$. Using the functional equation (1), $\bar{\omega}_s = w(s) + \alpha E_q f_q(U_q^{t+1})$. Since $f_q(U_q^{t+1}) \geq 0$, $\bar{\omega}_s \geq w(s)$, $s = 0, 1, \ldots S$. To show there is equality at state S substitute equation (4) into (2),

(7) $f_s'(U_s^t) = f_q'(U_q^{t+1})(1 + \psi_q^{t+1}) + \phi_q^{t+1}$ $\forall$ $q = 0, 1, \ldots S$.

But for $\omega_s = \bar{\omega}_s$, $U_s = \bar{U}_s$, where $f_s'(\bar{U}_s) \leq f_q'(\bar{U}_q) \leq f_q'(U_q^{t+1})$, for $q = 0, 1 \ldots S$. Therefore it must be optimal to have $U_q^{t+1} = \bar{U}_q$ for $q = 0, 1, \ldots S$. and hence $E_q f_q(U_q^{t+1}) = 0$, which implies $w(S) = \bar{\omega}_S$.

b. To show $\underline{\omega}_s \leq w(s)$ and $w(0) = \underline{\omega}_0$ consider again the functional equation (1). Then $u(\underline{\omega}_s) = u(w(s)) - \alpha E_q U_q^{t+1}$. As $E_q U_q^{t+1} \geq 0$, $\underline{\omega}_s \leq w(s)$. For $\omega_0 = \underline{\omega}_0$, $U_0 = 0$ and then from equation (7) it must be true that $U_q^{t+1} = 0$ and so $E_q U_q^{t+1} = 0$ from which $\underline{\omega}_0 = w(0)$.

This lemma shows that for any state s, in the optimum contract, the maximum possible wage in state s is never below the spot market wage, $w(s)$, and the minimum possible contract wage is never above it. In particular it is never optimal to pay a wage above the highest spot market wage and never optimal to pay a wage below the lowest. Therefore for each state $s = s_t$, $\omega(h_t) \in [\underline{\omega}_s, \bar{\omega}_s] \subseteq [w(S), w(0)] \subset [a, b]$. Lemma 7 also shows each interval $[\underline{\omega}_s, \bar{\omega}_s]$ may overlap or be disjoint from any other, but no interval contains any other. This result, makes it possible to derive the rule given in Theorem 2.

Theorem 2 presents the main result of this paper. It shows how optimal contract wages vary over time. It gives a simple rule whereby contract wages are updated from one period to the next. Suppose at date $t + 1$ the spot market wage is $w(s)$. The optimal contract wage is $\omega(h_t, s)$. At date t if the spot market wage was lower, denote this by $w(q) < w(s)$ and the associated contract wage by $\omega(h_{t-1}, q)$. If the spot market wage was higher, denote this occurence as state k, $w(k) > w(s)$, with the optimal contract wage $\omega(h_{t-1}, k)$.

THEOREM 2. For any date $t = 0, 1, \ldots \infty$, and any three states k, s, and q such that, $w(k) > w(s) > w(q)$ optimal contract wages at date $t + 1$ are:*

$$\omega(h_t, s) = \begin{cases} \omega(h_{t-1}, q) & \text{as } \omega(h_{t-1}, q) \geq \underline{\omega}_s, \\ \underline{\omega}_s & \phantom{\text{as }} \omega(h_{t-1}, q) < \underline{\omega}_s \end{cases} \qquad \forall\ q, s = 0, 1, \ldots, S,\ q < s$$

$$\omega(h_t, s) = \begin{cases} \omega(h_{t-1}, k) & \text{as } \omega(h_{t-1}, k) \leq \bar{\omega}_s, \\ \bar{\omega}_s & \phantom{\text{as }} \omega(h_{t-1}, k) > \bar{\omega}_s \end{cases} \qquad \forall\ k, s = 0, 1, \ldots, S,\ k > s.$$

$$\omega(h_t, s) = \omega(h_{t-1}, s) \qquad \forall\ s = 0, 1, \ldots, S.$$

PROOF: Using equations (3) and (7) it follows for any date t and any two states s_t and $s_{t+1} = s$,

$$\ell(\omega(h_{t-1}, s_t)) = \ell(\omega(h_t, s_{t+1}))(1 + \psi_s^{t+1}) + \phi_s^{t+1}$$

where $\ell(\cdot) = k^{-1}(\ )$ is continuous and strictly increasing. Then if $s_t = q$, $q < s$, $\psi_s^{t+1} = 0$ since $\bar{\omega}_q < \bar{\omega}_s$. So $\omega(h_{t-1}, q) = \omega(h_t, s) \geq \underline{\omega}_s$ unless $\phi_s^{t+1} > 0$, in which case $\omega(h_{t-1}, q) < \omega(h_t, s) = \underline{\omega}_s$. If $s_t = k$, $k > s$, $\phi_s^{t+1} = 0$, as $\underline{\omega}_k > \underline{\omega}_s$. So $\omega(h_{t-1}, k) = \omega(h_t, s_t)$ unless $\psi_s^{t+1} > 0$, when $\omega(h_{t-1}, k > \omega(h_t, s) = \bar{\omega}_s$. If $s_t = s$, then $\omega(h_{t-1}, s) = \omega(h_t, s)$, since $\omega(h_{t-1}, s) \in [\underline{\omega}_s, \bar{\omega}_s]$

Theorem 2 states that if at date $t + 1$ the spot market wage is higher than it was at date t, then the contract wage at date t + 1 is the same as at date t, or rises to the lowest level compatible with the new state. Similarly if the spot market wage at date $t + 1$ is lower than it was at date t, the contract wage at time $t + 1$ either remains unchanged or falls to the maximum level consistent with the lower spot market wage. Thus contract wages follow a simple Markov process. That is to say wages at date t + 1 depend only upon the spot market wage at that date and contract wages in the previous period.

Theorem 2 provides an easy method of solving for the optimal solution once the bounds $\underline{\omega}_s$ and $\bar{\omega}_s$ for each state s are known. These bounds depend on all the parameters of the model, but in Section 4 it will be shown there is no difficulty in principle in solving for $\underline{\omega}_s$ and $\bar{\omega}_s$ and a numerical example will be given. By Proposition 2 at date zero in any state s_0, $\omega(s_0) = \underline{\omega}_s$, since $U_s^0 = 0$. Therefore using Theorem 2 the solution is completely determined. Wages can take on at most $2(s + 1)$ possible values. $\{\bar{\omega}_s, \underline{\omega}_s, \ldots, \bar{\omega}_0, \underline{\omega}_0\}$. Then wages at any date t are just equal to wages at the last date in which one of the self-enforcing constraints was binding.

There is an obvious corollary to Theorem 2.

COROLLARY 2: At any date t, and for any two states k and q such that $w(k) > w(q)$, $\omega(h_{t-1}, k) \geq \omega(h_{t-1}, q)$.

Corollary 2 relates contract wages across states at any given date. Since each interval, $[\underline{\omega}_s, \overline{\omega}_s]$ contains the spot market wage and since $\bigcup_{s=0}^{S} [\underline{\omega}_s, \overline{\omega}_s] \subseteq [w(S), w(0)]$ contract wages are less flexible than spot market wages. Nevertheless for some date t and history up to that date h_{t-1} contract wages,may increase with the state whereas for another history, h'_{t-1} contract wages may be state invariant. Thus the state invariance result is history dependent. More will be said on this in the next section.

REMARK 9: If the firm is legally obliged to honour its contractual commitments but the worker is allowed to renege without sanction, (which is typically the case in the United Kingdom and elsewhere) then contract wages are completely downward rigid. To see this notice that since the contract is partially enforceable the firm can commit itself to pay a wage equal to b in every state, i.e. $\overline{\omega}_s = b$, $s = 0, 1, \ldots S$. Applying the rule derived in Theorem 2 contract wages will never be lowered at any date but will eventually rise to $\underline{\omega}_S$ with probability one. A contract which is legally binding on the employer will have higher expected profits than one which does not, but these increased profits are to be offset against the costs of the legal machinery.

3.4 Comparative Statics

This section examines how the optimal wage contract changes as the discount factor, α, varies. It is stressed this is a purely comparative static exercise: α is not allowed to vary over time. A comparative

static analysis could be undertaken for other parameters of the model such as the coefficient of the employees absolute risk aversion, or the variance of spot market wages, but this is unnecessary for the point to be made. By allowing α to vary it is illustrated how the relationships between the intervals $[\underline{\omega}_s, \overline{\omega}_s]$ are important in determining the structure of the optimal contract. This is accomplished in Proposition 3.

PROPOSITION 3: (i) For each state $s = 0, 1, \ldots, S$, $\overline{\omega}_s$ is non-decreasing and $\underline{\omega}_s$ is non-increasing in α; (ii) there exists an $\overline{\alpha} < 1$ such that for some $\alpha \in (\overline{\alpha}, 1)$, $\overline{\omega}_0 > \underline{\omega}_S$; (iii) there exists an $\alpha^* \in (0, 1)$ such that for $\alpha \in (0, \alpha^*)$, $\overline{\omega}_s = \underline{\omega}_s$, and for $\alpha \in (\alpha^*, 1)$, $\overline{\omega}_s > \underline{\omega}_s$ for each $s = 0, 1, \ldots, S$.

PROOF: (i) Since $f_s(\cdot)$ is continuous for each state $s = 0, 1, \ldots, S$ and $u(\cdot)$ is continuous, $\overline{\omega}_s$ and $\underline{\omega}_s$ are continuous functions of α. Consider first at time t a state s and the wage $\underline{\omega}_s$. From Theorem 2 and using equations (2) and (4) $U_q^{t+1} = 0$ for $q \geq s$; there is an $s' \in \{0, 1, \ldots, s\}$ such that $U_q^{t+1} = \overline{U}_q$, $f_q^-(\overline{U}_q) > f_s^+(0)$ for $q < s'$, $U_q^{t+1} < \overline{U}_q$, $f'_q(U_q^{t+1}) = f_q^+(0)$ for $s' < q < s$, and $f_{s'}(U_{s'}) \geq f_s^+(0)$; where $f_s^+(0)$ is the right hand derivative of $f_s(\cdot)$ at $U_s^t = 0$ and $f_{s'}^-(U_{s'})$ is the left hand derivative of $f_{s'}(\cdot)$ at $\overline{U}_{s'}$. There are two cases to consider: (a) $f_{s'}^-(\overline{U}_{s'}) > f_s^+(0)$. Consider a change in the discount factor from α to $\alpha + h$, $h \gtrless 0$. Then for h small enough by the continuity of $f_s(\cdot)$, $U_q^{t+1}(\alpha + h) = \overline{U}_q(\alpha + h)$ for $q \leq s'$. Then using equation (3) $f'_q(U_q^{t+1}) = -1/u(\underline{\omega}_s)$ for $s' < q < s$, hence $U_q^{t+1} = Z_q(\underline{\omega}_s)$ for $s' < q < s$, where $Z_q(\cdot)$ is strictly increasing and differentiable. From the definition of $\underline{\omega}_s$, $u(\underline{\omega}_s) - u(w(s)) + \alpha E_{q<s} U_q^{t+1} = 0$. Differentiating with respect to α, $(d\underline{\omega}_s/d\alpha)[U'(\underline{\omega}_s) + \alpha E_{s'<q<s} Z'_q(\underline{\omega}_s)] = -E_{q<s} U_q^{t+1} \leq 0$.

(b) If $f^-_{s'}(\bar{U}_{s'}) = f^+_s(0)$ then $U^{t+1}_{s'} = \bar{U}_{s'}$, if $h < 0$, and $U^{t+1}_{s'} = Z_{s'}(\underline{\omega}_s)$ if $h > 0$. Therefore $\underline{\omega}_s$ will not be differentiable at α, but since both the left hand and right hand derivatives are non-negative it still follows that $\underline{\omega}_s$ is non-increasing in α.

An analoguous argument can be made to show $\bar{\omega}_s$ is non-decreasing in α.

(ii) By Proposition 1 there exists an $\bar{\alpha} < 1$ such that for some $\alpha \in (0, \bar{\alpha})$ a constant wage contract is feasible. But by Theorem 2 a constant wage contract is only feasible if $\bar{\omega}_0 \geq \underline{\omega}_S$.

(iii) For $\alpha = 0$, $\bar{\omega}_s = \underline{\omega}_s = w_s$ for each state, the only feasible contract pays the spot market wage in every eventuality. Therefore for some small $\alpha > 0$ all the intervals $[\underline{\omega}_s, \bar{\omega}_s]$ will be disjoint. Then using Theorem 2 it follows that

$$u(\underline{\omega}_s) - u(w_s) + \alpha E_{q<s} \frac{u(\bar{\omega}_q) - u(\underline{\omega}_q)}{(1 - \alpha\rho_q)} = 0 ,$$

$$w_s - \bar{\omega}_s + \alpha E_{q>s} \frac{(\bar{\omega}_q - \underline{\omega}_q)}{(1 - \alpha\rho_q)} = 0 .$$

for each state, s. If for some k $\bar{\omega}_k > \underline{\omega}_k$ then either $\underline{\omega}_s \leq w_s < \bar{\omega}_s$ for $k > s$, or $\underline{\omega}_s < w_s \leq \bar{\omega}_s$ if $k < S$. Therefore if for some state k $\bar{\omega}_k > \underline{\omega}_k$ then for all other states $\bar{\omega}_s > \underline{\omega}_s$. Suppose $\bar{\omega}_s = \underline{\omega}_s = w_s$ for all s at $\alpha = \alpha^*$. It remains to show $\alpha^* > 0$. For $\bar{\omega}_q$ and $\underline{\omega}_q$ close to w_q the first equation given above can be linearized using a Taylor's series expansion

$$\underline{\omega}_s - w_s + \alpha E_{q<s} \frac{u'(w_q)}{u'(w_s)} \frac{(\bar{\omega}_q - \underline{\omega}_q)}{(1 - \alpha\rho_q)} = 0 .$$

This equation holds as an equality at $\alpha = \alpha^*$. Adding this equation to the second equation above

$$\bar{\omega}_s - \underline{\omega}_s = \alpha E_{q<s} \frac{u'(w_q)}{u'(w_s)} \frac{(\bar{\omega}_q - \underline{\omega}_q)}{1 - \alpha\rho_q} + \alpha E_{q>s} \frac{(\bar{\omega}_q - \underline{\omega}_q)}{(1 - \alpha\rho_q)} .$$

Let $\beta = (u'(w_0)/u'(w_s))(\alpha\rho_k/(1-\alpha\rho_k))$ where k is the mode state. Then it is possible, since $u'(w_0)$ and $u'(w_S)$ are positive and finite, to choose an $\alpha > 0$ such that $\beta < 1/S$. Next choose at state s' such that $\bar{\omega}_{s'} - \underline{\omega}_{s'} \geq \bar{\omega}_s - \underline{\omega}_s$ for all $s = 0, 1, \ldots, S$. Then at $\alpha = \alpha^*$

$$\bar{\omega}_{s'} - \underline{\omega}_{s'} = \alpha E_{q<s'} \frac{u'(w_q)}{u'(w_s)} \frac{(\bar{\omega}_q - \underline{\omega}_q)}{(1 - \alpha\rho_q)} + \alpha E_{q>s'} \frac{(\bar{\omega}_q - \underline{\omega}_q)}{1 - \alpha\rho_q}$$

$$\leq S\beta(\bar{\omega}_{s'} - \underline{\omega}_{s'}).$$

Therefore $\beta \geq 1/S$, which is a contradiction.

There are a number of important points to notice about this proposition. First for some $\alpha \in (0, \alpha^*)$ there is no feasible self-enforcing wage contract which dominates the spot market. In this case each interval $[\underline{\omega}_s, \bar{\omega}_s]$ contains just one point, the spot market wage. However even for these discount rates the rule derived in Theorem 2 still applies, albeit degenerately. The value of α^* depends on all the parameters of the model including the number of states, α^* will, ceteris paribus, fall as the number of states increases. This follows because the greater is the number of states the more flexibility there is to allocate potential gains in the future. Second at α^* each interval $[\underline{\omega}_s, \bar{\omega}_s]$ starts to expand, so some insurance is provided in every state. Each interval expands monotonically and from some $\bar{\alpha} \in (\alpha^*, 1)$ the intersection of all intervals is non-empty. Then from Theorem 2, for such a high enough discount factor, the wage $\underline{\omega}_S$ will be paid with probability one in the long run. This will be different from the constant wage contract unless the initial state at date zero was state S. This point has already been emphasized in Corollary 1. Third it is easy to see that for any discount factor α, the Markov chain described in, Theorem 2 will be ergodic. In the short run, however contract wages will not be stationary despite the stationarity of spot market wages. These points may perhaps become clearer with the help of an example.

4. Example

This section provides an illustration of the general principles set out in Section 3. This is done by assuming there are just two possible spot market wages w_0 and w_1. Algebraic expressions are then derived for the optimal contract wages as a function of the discount factor. The solution is contrasted with optimal stationary contracts which are more in the spirit of Bester [4]. It is shown precisely how these two solutions differ. A numerical example is given at the end of the section to show how simple it is to compute the optimal solution.

If there are just two states Proposition 3 shows that there are three regimes to consider depending upon the discount factor. Suppose α^* and α^{**}, $\alpha^{**} > \alpha^*$, are the critical discount factors which divide these regimes. Then for $\alpha \in (0, \alpha^*)$ there is no feasible contract which does not replicate the spot market solution; for $\alpha \in (\alpha^*, \alpha^{**})$, $\bar{\omega}_1 > \underline{\omega}_1 > \bar{\omega}_0 > \underline{\omega}_0$, so the two intervals $[\underline{\omega}_1, \bar{\omega}_1]$ and $[\underline{\omega}_0, \bar{\omega}_0]$ are disjoint; and for $\alpha \in (\alpha^{**}, 1)$, $\bar{\omega}_1 > \underline{\omega}_0 > \underline{\omega}_1 > \underline{\omega}_0$ so the two intervals overlap.

Consider first an $\alpha \in (\alpha^*, \alpha^{**})$; from Theorem 2, the evolution of wages in the optimal contract can be drawn as a tree diagram. This is done in Figure 1. Although the initial wages in Figure 1 are $\underline{\omega}_0$ and $\underline{\omega}_1$ a similar tree could be drawn if the initial wages were $\bar{\omega}_0$ and $\bar{\omega}_1$ (employee gets all the gain). Then by evaluating the expected profits and expected utility at the appropriate nodes of these trees it is easy to show

(8) $(1 - \alpha p_1)(w_0 - \bar{\omega}_0) - \alpha p_1(\underline{\omega}_1 - w_1) = 0,$

(9) $\alpha p_0(u(\bar{\omega}_0) - u(w_0)) - (1 - \alpha p_0)(u(w_1) - u(\underline{\omega}_1)) = 0.$

and $\underline{\omega}_0 = w_0$, $\bar{\omega}_1 = w_1$. Looking at Figure 1 it is apparent that $\bar{\omega}_1$ is never paid in the optimal contract. Similarly $\underline{\omega}_0$ is only paid in state zero if state one has not occured before. Therefore, in the long run, the optimal contract pays wages $\bar{\omega}_0$ and $\underline{\omega}_1$ with probabilities p_0 and p_1 respectively. In the long run the expected contract wage bill is $p_0 \bar{\omega}_0 + p_1 \underline{\omega}_1$ in each period. Using equation (8) this is less than the expected wage bill of a firm that hires on the spot market. Similarly from equation (9), the long run, per period expected utility the employee gets is $p_0 u(\bar{\omega}_0) + p_1 u(\underline{\omega}_1)$ which is greater than the expected utility to be derived from selling labour on the spot market. That is to say although at date zero the worker is indifferent between having or not have a contract, at date $t = 1, 2, \ldots \infty$ the employee can expect to strictly prefer remaining on the contract rather than selling labour on the spot market.

Suppose next, $\alpha \in (\alpha^{**}, 1)$, in which case the two intervals overlap. The relevant tree diagram is drawn as Figure 2. It is still true that $\underline{\omega}_0 = w_0$ and $\bar{\omega}_1 = w_1$ but for $\alpha \in (\alpha^{**}, 1)$, $\underline{\omega}_1$ and $\bar{\omega}_0$ are determined by

(10) $\bar{\omega}_0 = (1 - \alpha p_1)w_0 + \alpha p_1 w_1$

(11) $u(\underline{\omega}_1) = \alpha p_0 u(w_0) + (1 - \alpha p_1)u(w_1).$

Equations (10) and (11) are derived in the same way as equations (9) and (10), i.e. by evaluating expected utility and profits at nodes on the tree diagram. It should be stressed that this procedure is perfectly general. It can be applied to any finite number of states. The solution can then easily be worked out on a computer. From Figure 2 it is apparent that $\underline{\omega}_1$ is paid unless state zero occurs and state one has not occured before in which case $\underline{\omega}_0$ is paid. In the long run, $\underline{\omega}_1$ is paid with probability one. In this sense for α near enough one the optimal contract is a constant wage contract. For $\alpha < 1$, $\underline{\omega}_1$ is higher than the certainty equivalent of spot market wages. The firm's expected wage bill is $\underline{\omega}_1$ in the long run and is less than the expected wage bill from hiring in the spot market. The employees expected long run utility is also higher than the spot market equivalent.

These results can be further illuminated by comparing the optimal contract with an optimal stationary contract. A stationary contract is a contract in which the distribution of wages across states is the same at each date. An optimal stationary contract is a stationary contract which maximizes the firms expected discounted profits subject to the feasibility constraints. So if there are just two states a stationary contract specifies two wages, ω_0 if state zero occurs and ω_1 otherwise. Clearly the only stationary contract which is self-enforcing and offers the employee a zero net gain is the spot contract, $\omega_0 = w_0$, $\omega_1 = w_1$. Therefore a stationary contract will offer the employee a positive net gain. This is essentially the methodology adopted by Bester [4] although he does not consider the firms self-enforcing constraints.

If the wage contract is restricted to be stationary, the self-enforcing constraints for the firm and worker are the same at each date. As there are just two states, the firm is only constrained in state zero when the contract wage is above the spot wage and the worker is only constrained in state one when the contract wage is below the spot wage. In fact for a stationary contract the firm's self-enforcing constraint is that the left hand side of equation (8) should be non-negative when ω_0 replaces $\bar{\omega}_0$ and ω_1 replaces $\underline{\omega}_1$. Similarly, with the same substitutions the

employee's self-enforcing constraint is that the left hand side of equation (9) should be non-negative. It is clear that the worker's constraint always binds, since otherwise the employer could increase profits without violating feasibility. Equally since the employer is risk neutral it is never optimal for $\omega_0 > \omega_1$. Therefore for $\alpha \in (\alpha^{**}, 1)$ the firm's constraint does not bind and the optimal stationary contract is $\omega_0 = \omega_1 = \omega = \underline{\omega}_1$; and for $\alpha \in (\alpha^*, \alpha^{**})$, there is a solution in which $\omega_0 < \omega_1$ and both constraints bind, it is $\omega_0 = \bar{\omega}_0$, $\omega_1 = \underline{\omega}_1$. For $\alpha \in (0, \alpha^*)$ there is no feasible contract, stationary or otherwise.

In the two state case the optimal stationary contract differs from the optimal contract in one respect. When state zero occurs and state one has not previously arisen, the optimal contract pays the spot market wage w_0 whereas the optimal stationary contract pays $\omega_0 > w_0$. The optimal contract offers no insurance to the employee against an initial sequence of bad states. In the long run the optimal contract will be stationary and for $\alpha \in (\alpha^{**}, 1)$ offer complete insurance but it is not quite clear how this can compensate for an initial run of bad luck.

To put some flesh on these bones and illustrate how easy it is to compute the optimal solution consider the following example.

EXAMPLE: Let $u(\omega) = \sqrt{\omega}$, $w_0 = 1$, $w_1 = 4$, $p_0 = p_1 = \frac{1}{2}$. Then $\alpha^* = 2\sqrt{2} - 2$ $\alpha^{**} = 7 - \sqrt{37}$.

The optimal interval endpoints are plotted against α in Figure 3. For $\alpha \in (0, \alpha^*)$ the optimal contract wages are just the spot market wages. At α^* the optimal wages $\bar{\omega}_0$ and $\underline{\omega}_1$ start to rise and fall respectively. At α^{**}, $\underline{\omega}_1 = (23 - 3\sqrt{37})/2$. For $\alpha \in (\alpha^{**}, 1)$ the optimal wage $\underline{\omega}_1$ continues to fall tending, as α tends to one, to the certainty equivalent wage $9/4$.

5. Concluding Remarks

This paper has shown that an optimal self-enforcing wage contract exists (Theorem 1) and how the optimal solution can be calculated (Theorem 2). It has been shown how optimal contract wages are more inflexible than spot market wages.

The model was extensively discussed in Section 2. Therefore the concluding remarks will be confined to possible extensions. There are at least two possible extensions which seem worthwhile to pursue. First there is the question of finitely lived workers. As an assumption infinitely lived firms is just about tenable but infinitely lived workers is certainly implausable. However with infinitely lived firms and finitely lived workers a self-enforcing contract may still exist since the firm will wish to maintain its reputation over generations. In fact it is to be expected that contracts in this context will exhibit much more of Holmstrom's 'front-loading' solution.

Perhaps the most interesting extension is to the asymmetric information case. Suppose the spot market wage is interpreted as a random opportunity cost which can only be observed by the employee. Then in a single period context the only feasible contract has a constant wage independent of the employer's preferences. But in an infinite horizon model it must be possible to do better than the single period game, since wages in the future can be conditioned upon actions in the past. The framework given above should prove useful in addressing this problem.

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Footnotes

1. We are very grateful to Frank Hahn and members of his Quaker workshop group for helpful and encouraging comments. The financial support of the Economic and Social Research Council is much appreciated.
2. See e.g. Azarindis [1], Bailey [2], and Gordon [5].
3. See e.g. Hart [7] and the Quarterly Journal of Economics Symposium [12].
4. See e.g. Grossman, Hart and Maskin [6], Hahn [9], Newbery and Stiglitz [11], and Thomas [14].

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FIGURE 2 - $\alpha \in (\alpha^{**}, 1)$

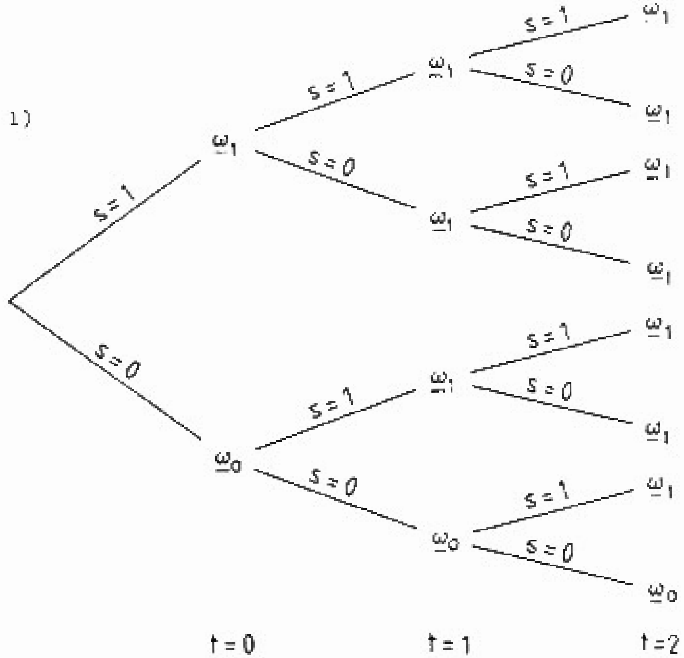
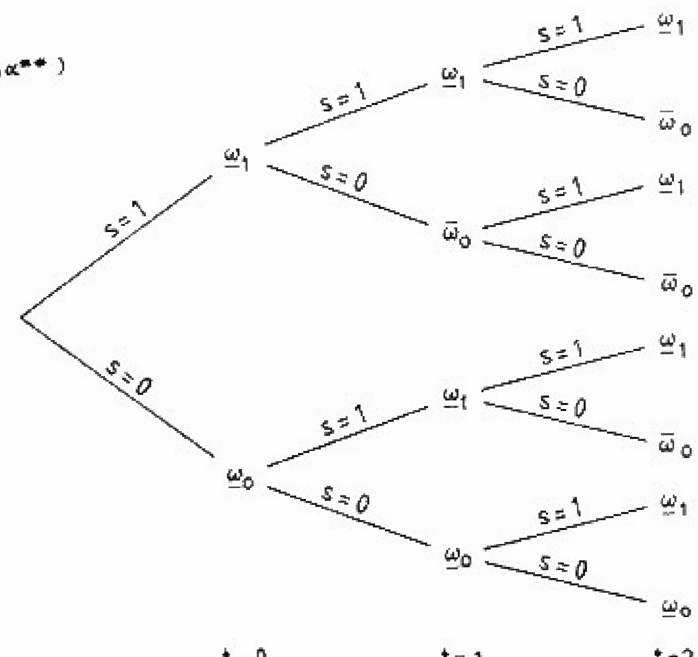


FIGURE 1 - $\alpha \in (\alpha^{*}, \alpha^{**})$



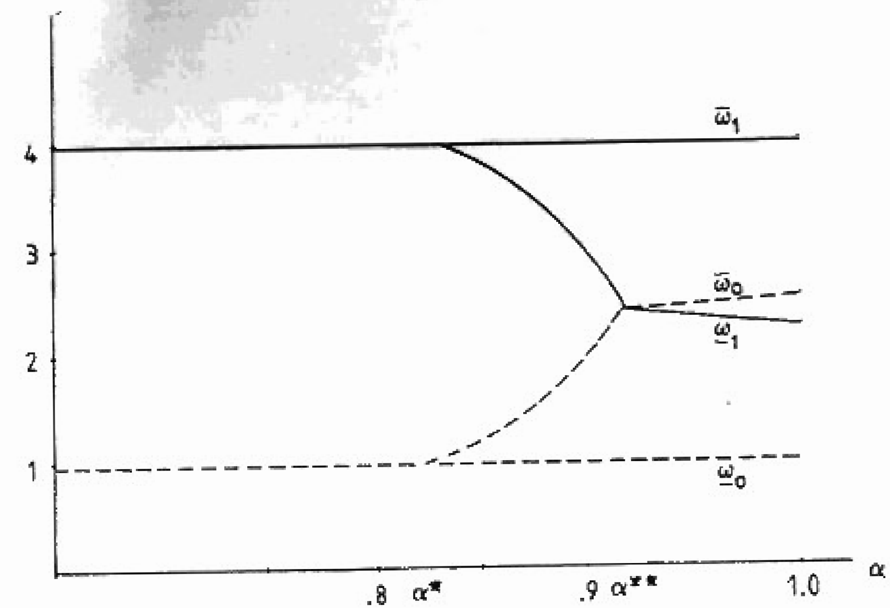


FIGURE 3

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