Sonderforschungsbereich 178
„Internationalisierung der Wirtschaft“

Diskussionsbeiträge

Universität Konstanz

Juristische Fakultät

Fakultät für Wirtschaftswissenschaften und Statistik

Jonathan Thomas and
Timothy Worrall

Income Transfers to LDC's under Asymmetric Information: a Two Country Model

Postfach 5560
D-7750 Konstanz

Serie II — Nr. 38
November 1987

INCOME TRANSFERS TO LDC'S UNDER ASYMMETRIC INFORMATION: A TWO COUNTRY MODEL

Jonathan Thomas* and Timothy Worrall

Serie II - Nr. 38

November 1987

*) Visiting from Department of Economics,
University of Warwick, Coventry, U.K.

1. Introduction

We consider the problem of stabilising the income of a country, henceforth the borrower, which would always prefer a stable income to a random income with the same average, i.e. a country which is risk-averse. This would be appropriate for countries which are heavily reliant upon agricultural output, or those whose exports are concentrated in a few markets with prices which fluctuate considerably. We suppose that there is another country, or possibly an international organisation, which is risk-neutral and has access to relatively large financial resources. We shall refer to this country as the lender. We further suppose that there is asymmetric information: the lender cannot observe the borrower's income. This is extreme though not that unrealistic in many cases.

If there is more than one period a simple loan scheme can provide some insurance (Grossman, Levhari and Mirman [5], Yaari [11]). Townsend [10] has shown that it is possible to do better than these simple loan schemes by using efficient contracts. Our purpose is to examine the properties of efficient loan contracts for any time horizon and any discount factor.

The efficient contract corresponds to the solution of a dynamic programming problem. So it can be calculated in a straightforward recursive manner. We can show directly that the second-best Pareto-frontier converges to the first-best frontier as the discount factor tends to one. So the efficient contract nearly provides the first-best utilities if the time horizon is sufficiently long and the discount factor is sufficiently high (Section 7).

If the contract is fully enforeable and the time horizon is infinite then the borrower's utility becomes arbitrarily negative with probability one. This could be interpreted as saying that a debt crisis will develop with probability one. It also suggests that the contract will become difficult to enforce. Nevertheless even if the borrower is not legally bound by the contract it will still not break down (Section 6). In the special case where the borrower's utility function is exponential the contract transfers consumption between any two states at a constant rate of interest which is less than the rate of time preference (Section 8). This can be interpreted as saying that soft loans are optimal under these circumstances.

2. The Model

There is a borrower and a lender. Both live $T+1$ periods and can transfer between themselves a single non-storable consumption good called income. Both maximize lifetime utility and discount the future by a common factor $\alpha \in (0, 1)$.

ASSUMPTION 1. *The per period utility function of the borrower is* $v : (a,\infty) \rightarrow \mathbf{R}^1$ $\sup\ v(c) < \infty$, $\inf v(c) = -\infty$, *v is* C^2 *with* $v'>0$, $v''<0$, $-v''/v'$ *non-increasing and* $\lim_{c \rightarrow -\infty} v'(c) = \infty$.

ASSUMPTION 2. *At each date* $t=0,1,\ldots,T$ *the borrower has a random income* s_t *which is identically and independently distributed over a finite set* $S = \{\theta_1, \theta_2, \ldots, \theta_S\}$.

The lender is risk neutral, cannot observe past or present income but can monitor or control all the borrower's transactions[1]. Most loan contracts do have indenture restrictions which monitor or control the borrower's transactions. For example bank loans require other debts to be disclosed and insurance contracts are void if material facts are unreported.

It is in the interest of the borrower and lender to negotiate a contract to stabilise the borrower's consumption. For the moment we will assume the contract is legally enforceable but will relax the assumption in Section 6.

DEFINITION 1. *A loan contract* b^{T+1} *is a sequence of functions* $(b_t)_{t=0,1,\ldots,T}$; *where each* $b_t : S^{t+1} \rightarrow (a-s_t,\infty)$.

Thus a loan contract is a sequence of payments as a function of history, $h^t = (s_0, s_1, \ldots s_t) \in S^{t+1}$. As a convention $b(h^t)$ is positive if payment is made to the borrower and is negative if there is a repayment.

By the Revelation Principle there is no loss in restricting attention to incentive compatible contracts where the borrower is deterred from reporting a false income[2]. Let V_i be the highest expected future utility (discounted to date $t + 1$) which the borrower can get if he reports his income at date t is $r_t = \theta_i$. Since income is *i.i.d.* V_i depends only upon the reported history, $k^t = (r_0, r_1, \ldots, r_t)$ and not upon the actual history, $h^t = (s_0, s_1, \ldots, s_t)$. Let $b_j = b_t(k^{t-1}, \theta_j)$ be the amount borrowed at time t if $r_t = \theta_i$ and the past reported history is k^{t-1}. Then if b^{T+1} is incentive compatible,

$$v(b_i + \theta_i) + \alpha V_i \geq v(b_j + \theta_i) + \alpha V_j \qquad (1)$$

for every k^{t-1} and all i,j. We let B be the set of incentive compatible contracts.

3. The Dynamic Programming Characterisation of Efficient Contracts

An efficient contract is incentive compatible and undominated by any other incentive compatible contract. Efficient contracts can be characterized by a dynamic program. The basic idea is simple : in an efficient contract after any history h^t, $s_t = \theta_i$, the remaining part of the contract from date $t + 1$ onwards must itself be efficient. Otherwise replacing this part of the contract by an efficient contract which gives the borrower the same expected future utility, V_i, will make the lender better off. Since income is *i.i.d.*, V_i does not depend on actual history so the new contract will be incentive compatible for every possible h^t. Moreover, the contract payments at time t and the associated V_i's must be chosen to maximise the lender's utility subject to being incentive compatible at t and giving the lender a fixed amount of utility from t onwards. This dynamic programming characterisation implies that there is no incentive to renegotiate the contract at any date, since any renegotiation would make at least one party worse off.

Let b^k be a contract with k periods remaining and let

$$B(V) = \{b^k \in B : E \sum_{\tau=0}^{k-1} \alpha^{\tau}(v(b(h^{\tau}) + s_{\tau}) - v(s_{\tau})) = V\},$$

be the set of incentive compatible contracts giving the agent a net gain of V relative to autarky.

DEFINITION 2. The *value* function with k periods to go (at time T-k+1) is

$$U_k(V) = \sup_{b_k \in B(V)} - E \sum_{\tau=0}^{k-1} \alpha^{\tau} b^k(h^{\tau})$$

for any $V \in (-\infty, d_k)$, where $d_k = (\sup v(c) - \Sigma_{i \in S}\, v(\theta_i))(1-\alpha^k)/(1-\alpha)$.

For $T = \infty$ the value function, which is independent of time, is denoted $U_*(V)$. The value function gives the most utility the lender can achieve when the borrower gets a net utility gain of V. $U_k(V)$ is not defined to the right of d_k. However, let $U_k(V) = -\infty$ for $V \geq d_k$ so each U_k is defined on the common interval $(-\infty, d_\infty)$ and takes values in the extended reals.

DEFINITION 3. For any function $U : (-\infty, d_\infty) \to R^1 \cup \{-\infty\}$ the *one-step operator L* satisfies

$$L(U)(V) = \sup_{(b_i, V_i)_{i \in S} \in \Lambda(V)} \Sigma_{i \in S} \pi_i(-b_i + \alpha U(V_i)),$$

$$\Lambda(V) = \{ (b_i, V_i)_{i \in S} : b_i \in (a - \theta_i, \infty), V_i \in (-\infty, d_\infty) ; \Sigma_{i \in S} \pi_i(v(b_i + \theta_i) - v(\theta_i) + \alpha V_i) = V,$$
$$v(b_i + \theta_i) + \alpha V_i \geq v(b_j + \theta_i) + \alpha V_j \text{ for all } i, j \}.$$

For $k \geq 1$ the one-step operator L defines the value functions recursively through the optimality equation

$$U_k(V) = L(U_{k-1})(V). \quad (2)$$

It is proved in Lemma 1 (Appendix). In the finite horizon problem U_k is found by backwards induction, starting with $U_0(V) = 0$ while U_* is a fixed point of L.

It would be helpful for both analytical and computional reasons to know if $U_* = \lim_{k \to \infty} U_k$. To use standard arguements it will be necessary to restrict the space of functions. There are natural bounds on U_k. It is at least as good as the (incentive compatible) contract which pays a constant amount y_k at all dates, where y_k satisfies $\Sigma_{i \in S} \pi_i(v(y_k + \theta_i) - v(\theta_i))(1 - \alpha^k)/(1 - \alpha) = V$. But it is no better than the unconstrained first best contract which pays $c_k - s_t$ at all dates, where c_k satisfies $\Sigma_{i \in S} \pi_i(v(c_k) - v(\theta_i))(1 - \alpha^k)/(1 - \alpha) = V$. Thus

$$-(1-\alpha^k)\,y_k/(1-\alpha) \leq U_k(V) \leq (1-\alpha^k)\,\Sigma_{i\in S}\pi_i(\theta_i - c_k)/(1-\alpha) \tag{3}$$

$$-y_\infty/(1-\alpha) \leq U_*(V) \leq \Sigma_{i\in S}\pi_i(\theta_i - c_\infty)/(1-\alpha). \tag{4}$$

These bounds tie down U_k quite tightly. If $a = -\infty$ then $lim_{V\to-\infty}U_k(V) = \infty$, while it is finite if $a > -\infty$. Similarly $lim\ U_k(V) = -\infty$ as $V \to d_k$. This is depicted in Figure 1.

Let F be the space of continuous functions on $(-\infty, d_\infty)$ lying between the bounds in (4). It is complete in supremum metric (Lemma 2) and by standard arguments L is a contraction in F. So U_* is the unique fixed point of L in F and for any $U \in F$, $lim_{k\to\infty}L^k(U) = U_*$. To show that U_* is the limit of the finite horizon value functions is not straightforward as $U_k \notin F$. Nevertheless Lemma 3 shows $lim_{k\to\infty}U_k$ is a fixed point of L and since the limit belongs to F (take limits in (3)) it follows that $lim_{k\to\infty}U_k = U_*$.

4. The Efficient Contract

To calculate the efficient contract it is necessary to solve the prgramming problem defined by equation (2). It is straightforward to see that $U_1(V)$ is strictly concave, therefore we will invoke the induction hypothesis and assume $U_{k-1}(V)$ is strictly concave too. Standard arguments then imply that there exists a unique solution to equation (2). To proceed rewrite equation (1) as $C_{i,j} = v(b_i + \theta_i) - v(b_j + \theta_i) + \alpha(V_i - V_j) \geq 0$. It assumes the borrower wishes to report the true income θ_i rather than the false income θ_j. Since v is concave, if the local constraints $C_{i,i-1} \geq 0$ and $C_{i,i+1} \geq 0$ hold for each $i \in S$ then the global constraints $C_{i,j} \geq 0$ hold for each $i,j \in S$.[3] It follows from the concavity of v and adding $C_{i,i-1} \geq 0$ and $C_{i-1,i} \geq 0$ that $b_{i-1} \geq b_i$ and $V_i \geq V_{i-1}$. Thus in low income states more is borrowed at the cost of lower future utility. In Lemma 4 it is shown that the local upward incentive compatibility constraints, $C_{i,i+1} \geq 0$, never bind but the local downward incentive compatibility constraints, $C_{i,i-1} \geq 0$, always do, and the solution has coinsurance, that is $-b_i + \alpha U(V_i) \geq -b_{i-1} + \alpha U(V_{i-1})$ and $v(b_i + \theta_i) + \alpha V_i > v(b_{i-1} + \theta_{i-1}) + \alpha V_{i-1}$. Under Assumption 1 it is then possible to complete the induction argument by showing that U_k is also strictly concave.

PROPOSTION 1. *$U_k(V)$ is decreasing, strictly concave and continuously differentiable on $(-\infty, d_k)$ and $U_*(V)$ is decreasing, concave and continuously differentiable on $(-\infty, d_\infty)$.*

Letting λ, $(\mu_i)_{i\in S}$, be the multipliers associated with the respective constraints $\Sigma_{i\in S}\pi_i\,(v(b_i + \theta_j) - v(\theta_i) + \alpha V_i) = V$. $C_{i,i-1}\geq 0$, the first order conditions

$$\pi_i(1-\lambda v'(b_i+\theta_i))=\mu_i v'(b_i+\theta_i) - \mu_{i+1} v'(b_i+\theta_{i+1}) \tag{5}$$

$$\pi_i(U_{k-1}'(V_i)+\lambda) = \mu_{i+1} - \mu_i \tag{6}$$

for $i=2,3,\ldots,S$, where $\mu_1 = \mu_{S+1} = 0$, are both necessary and sufficient. There is also an envelope condition

$$U_k'(V) = -\lambda. \tag{7}$$

Since V incorporates all the information necessary to calculate $(b_i,V_i)_{i\in S}$, and $U_k(V)$ is strictly concave the efficient contract can be determined recursively using equations (5), (6) and (7) starting with the intitial value V_0. It is unique and coinsures both borrower and lender (Lemma 4). These results are summarized by Proposition 2:

PROPOSTION 2. *For any T there exists a unique efficient coinsurance contract such that after any history $b_i\leq b_{i-1}$, $V_i\geq V_{i-1}$, $i=2,3,\ldots,S$. The local upward incentive compatibility constraints never bind but the local downward incentive compatibility constraints always do.*

5. The Infinite Horizon Contract.

One way to increase the borrower's utility by a unit is to increase every V_i by a factor of $1/\alpha$ and keep every b_i constant. Such a change preserves incentive compatibility at a cost to the lender of $\Sigma_{i\in S}\pi_i U_{k-1}'(V_i)$. By the envelope theorem this is locally as good a way to increase V as any other and so is equal to $U_k'(V)$. Viewed as a stochastic process $U_k'(V)$ is a martingale, and using the martingale convergence theorem,

PROPOSITION 3. *If* $T = \infty$, V_t *converges to* $-\infty$ *almost surely.*

Intuitively the advantage of having a history dependent contract stems from using future utility, the V_i's, as inducements to truth-telling. To do this the V_i's must be different. Since U_* is concave this is costly because the lender's future utility falls as the dispersion of the V_i's increases. To obtain incentive compatibility when V is close to d_∞ is very costly while if V is arbitrarily negative U_* is nearly flat and the cost is comparatively small. A contract in which V declines over time can induce the borrower to tell the truth by using large variability in future utility and at the same time smooth consumption in the initial periods.

Since V tends to decline the borrower's incentive to abide by the contract also declines. Therefore, in the next section we examine self-enforcing contracts in which the borrower never has an incentive to renege.

6. Self-Enforcing Loan Contracts

Assuming the contract is legally enforceable helps to determine its long-run properties but there are certainly circumstances in which either borrower or lender can renege without legal penalty. An obvious example is international loans where the debtor country can often repudiate its debt without incurring legal sanctions. An efficient contract must take this possibility into account. What is less clear is whether or not the contract might terminate before the last date T.

Suppose only the borrower can renege but has no outside opportunities. To prevent him from reneging after he has observed his income the contract must satisfy for all $i \in S$

$$v(b_i + \theta_i) + \alpha V_i \leq v(\theta_i). \tag{8}$$

This *ex post* constraint also implies that the corresponding *ex ante* constraint holds. The efficient contract is found by adding (8) to the definition of the one step operator, L.

We shall show the efficient contract never terminates. A contract terminates if payments are always zero after some particular history (though we consider only histories after which at least two periods remain since zero payments in the final period in all states is quite possible). Suppose to the contrary that the contract does terminate at some point. Consider reducing the payment in state S from zero by a small amount Δb_S and increasing all payments next period by Δb so that

$$v(-\Delta b_S + \theta_S) + \alpha \Sigma_{i \in S} \pi_i v(\Delta b + \theta_i) = v(\theta_S) + \alpha \Sigma_{i \in S} \pi_i v(\theta_i).$$

By the concavity of v this change is incentive compatible and it satisfies (8). If Δb_S is small enough $\Delta b_S / \Delta b \approx \alpha \Sigma_{i \in S} \pi_i v'(\theta_i)/v'(\theta_S) > \alpha$, so the gain to the lender is $\pi_S(\Delta b_S - \alpha \Delta b) > 0$.

Thus the change improves the lender's utility and leaves the borrower's utility constant. Hence by the principle of optimality the termination could not be part of an efficient contract.

PROPOSITION 4. *If the borrower can renege and has no outside opportunities then an efficient contract will not terminate.*

7. The Efficient Contract for Large Discount Factors.

In repeated hidden-action models the first-best utilities can be approached as the discount factor gets close to unity. Radner [9] uses a statistical approach based on a contract which penalises the agent periodically if his record fails a review test. Fudenberg, Holmstrom and Milgrom [3] use the idea that the agent can smooth his own consumption by borrowing and saving at a fixed rate of interest equal to the rate of time preference. While neither of these contracts is generally second-best efficient, they are good enough in the limit.

It is, however, simple to prove directly that the second-best Pareto-frontier converges to the first-best frontier[4]. To keep utility bounded multiply by $(1-\alpha)$. Let V_i^α be the borrower's normalised discounted future utility and $U^\alpha(V_i^\alpha)$ the corresponding utility of the lender. For any given V the first-best contract gives the lender an expected return of $-\Sigma_{i\in S}\pi_i b_i^*$, where $v(b_i^* + \theta_i) = V$ for all $i \in S$. It is necessary to show $\lim_{\alpha\to 1} U^\alpha(V) = -\Sigma_{i\in S}\pi_i b_i^*$. Consider a contract which pays b_i^* in the first period, satisfies the downward incentive constraints with equality and is efficient thereafter. Because it is incentive compatible it cannot offer more utility than the efficient contract: $U^\alpha(V) \geq -\Sigma_{i\in S}\pi_i b_i^* + (\alpha/(1-\alpha))\Sigma_{i\in S}\pi_i h_\alpha(V_i^\alpha)$, where $h_\alpha(V_i^\alpha) \equiv U^\alpha(V_i^\alpha) - U^\alpha(V)$. So it is necessary to show that $(\alpha/(1-\alpha))\Sigma_{i\in S}\pi_i h_\alpha(V_i^\alpha) \to 0$ as $\alpha \to 1$.

PROPOSITION 5. *For given V, as* $\alpha \to 1$ *the utility of the lender from the efficient contract tends to the first-best level.*

The intuition is straightforward: for α close to unity incentive compatibility can be attained by a small divergence in the V_i^α's. The cost of this divergence is $\Sigma_{i\in S}\pi_i h_\alpha(V_i^\alpha)$ which is positive because U^α is concave. It goes to zero faster than α goes to unity because U^α is differentiable and hence locally linear.

By the convergence of the finite horizon value functions to the infinite horizon value functions (Lemma 2 & 3) we have

COROLLARY. *For given V, and any* $\varepsilon > 0$, *there is an* α' *and a* T' *such that* $U_T^{\alpha}(V) \geq \Sigma_{i \in S} \pi_i b_i^* - \varepsilon$ *for* $T > T'$, $\alpha > \alpha'$.

By Proposition 3, the efficient contract cannot converge uniformly to the first-best (constant consumption) contract. Nevertheless pointwise convergence can be proved:

PROPOSITION 6. *For any history* h^t *the efficient contract payments converge to their first-best levels as* $\alpha \rightarrow 1$.

8. Constant Absolute Risk Aversion

A special case of Assumption 1 is the constant absolute risk aversion utility function $v(c) = -exp(-Rc)$ where $v: (-\infty, \infty) \rightarrow (-\infty, 0)$. With an infinite time horizon the efficient contract has a particularly simple form.

PROPOSITION 7. *If* $T = \infty$ *and* $v(c) = -exp(-Rc)$ *then at the optimum* $exp(-Rb_i) = -a_iV$, $V_i = d_iV$, *where the* a_i *and* d_i*'s are constant satisfying* $a_i \geq a_{i-1} > 0, d_{i-1} \geq d_i > 0$; $\Sigma_{i \in S} \pi_i d_i^{-1} = 1$, $\Sigma_{i \in S} \pi_i a_i^{-1} = \Sigma_{i \in S} \pi_i c_i / (1-\alpha)$, *and* $U_*(V) = R^{-1}(1-\alpha)^{-1} \{log(-V) + \Sigma_{i \in S} \pi_i log\, a_i + \alpha(1-\alpha)^{-1} \Sigma_{i \in S} \pi_i log d_i\}$.

Payments depend on the number of times each state occurs but not on the order in which they occur. So if τ_j is the number of times state j occurs in the history h^t and V_o is the borrower's initial utility gain, then from Proposition 7 the payment in state i at date $t+1$ is

$$b(h^t, \theta_i) = -R^{-1}\{log\, a_i + \Sigma_{j \in S} \tau_j log\, d_j + log\,(-V_o)\}.$$

Not only do repayments tend to infinity almost surely, but expected repayments rise each period, becoming arbitrarily large.

Suppose that at some date state j is announced instead of state i, $j < i$, so an extra payment is received from the lender. Because the order of announcement doesn't matter, this can be corrected at a later date by announcing i instead of j. In the meantime the borrower will be paying back more than he otherwise would, and so an implicit rate of interest between states i and j is

$$r_{ij} = -\log(d_i/d_j)/\log(a_i/a_j)$$

where $0 \leq r_{ij} < (1-\alpha)/\alpha$. It must be positive to ensure incentive compatibility and less than the rate of time preference to allow the borrower to shed risk: he borrows cheaply but earns a low rate of return on savings. The interest rate is constant because of the absence of income effects (see also Fudenberg, Holmstrom and Milgrom [3]). In the limit as $\alpha \to 1$ each $r_{ij} \to 0$ so consumption risk can be eliminated (Yaari [11]). These results are summarised by

PROPOSITION 8. *If* $T = \infty$ *and* $v(c) = -\exp(-Rc)$ *then (i) there are* S-1 *implicit rates of interest;* $0 \leq r_{ij} < (1-\alpha)/\alpha$, *for each* $i,j \in S$; *(ii) expected repayments increase over time and (iii) repayments become positive in every state with probability one.*

Appendix

LEMMA 1. *$U_k(V)=L(U_{k-1})(V)$ for $V\in(-\infty,d_\infty)$.*

Proof. *(i)* We first show $U_k(V)\leq L(U_{k-1})(V)$. Define

$$U[b^k]= -E\sum_{\tau=0}^{k-1} \alpha^{\tau}b(h^{\tau}), \qquad U[b^k:h^t]= -E[\sum_{\tau=t+1}^{k-1} \alpha^{\tau-t-1}b(h^{\tau}):h^t]$$

$U[b^k]$ is the net gain to the lender from the contract b^k and $U[b^k:h^t]$ is the net gain after the history h^t. Define $V[b^k]$ and $V[b^k:h^t]$ analogously.

So for any $V\,(-\infty,d_\infty)$ and any $b^k\in B(V)$, $U[b^k]= \sum_{i\in S} \pi_i(-b_i+ U[b^k: s_0=\theta_i])$. Then by the definition of U_{k-1}, $U[b^k: s_0]\leq U_{k-1}(V[b^k: s_0])$ and since $(b_i, V[b^k:s_0=\theta_i])_{i\in S}\in\Lambda(V)$, $\sum_{i\in S} \pi_i(-b_i+ U_{k-1}(V[b^k: s_0=\theta_i])\leq L(U_{k-1})(V[b^k])$. Therefore taking the supremum over all $b^k\in B(V)$, $U_k(V)= \sup U[b^k]\leq L(U_{k-1})(V)$.

(ii) We now show $U_k(V)\geq L(U_{k-1})(V)$. There exists some $(\beta_i,\vartheta_i)_{i\in S}\in\Lambda(V)$ and $\varepsilon>0$ such that $\sum_{i\in S}\pi_i(-\beta_i+ \alpha U_{k-1}(\vartheta_i))\geq L(U_{k-1})(V)-\varepsilon$ for any $V\in(-\infty, d_\infty)$. Equally $U[b^k: s_0=\theta_i]\geq U_{k-1}(\vartheta_i)-\varepsilon$, where $[b^k: s_0=\theta_i]\in B(\vartheta_i)$. Let β^k be the contract which pays β_i in the first period and follows $[b^k: s_0=\theta_i]$ thereafter. Since income is *i.i.d.* $\beta^k\in B(V[\beta^k])$. So $U[\beta^k]\geq\sum_{i\in S}(-\beta_i+\alpha U_{k-1}(\vartheta_i))-\alpha\varepsilon\geq L(U_{k-1})-(1+\alpha)\varepsilon$. Since ε is arbitrary, taking the supremum over all $b^k\in B(V)$, $U_k(V)\geq L(U_{k-1})(V)$.

LEMMA 2. *F is complete in the supremum metric.*

Proof. It suffices to show the gap between the bounds in (4) is itself bounded. Since v is increasing and $\sum_{i\in S}\pi_i v(y_\infty+\theta_i)=v(c_\infty)$, for given V, $y_\infty+\theta_1\leq c_\infty$ and so $y_\infty+\theta_i+\theta_1 \leq y_\infty+\theta_S+\theta_1\leq c_\infty+\theta_S$. Thus $y_\infty+\theta_i-c_\infty\leq\theta_S-\theta_1$ and therefore it must be the case that $y_\infty+\sum_{i\in S}\pi_i(\theta_i-c_\infty)/(1-\alpha)\leq(\theta_S-\theta_1)/(1-\alpha)$.

LEMMA 3. *$Lim_{k\to\infty}U_k=L(Lim_{k\to\infty}U_k)$.*

Proof. Define $U_\infty=Lim_{k\to\infty}U_k$. It is obvious the lender can do at least as well in $k+1$ periods as he can in k periods. So for any V, $U_0\leq LU_0\leq L^2U_0\leq\ldots\leq L^kU_0\leq\ldots\leq U_\infty$. Hence $L^{k+1}U_0\leq LU_\infty$ and taking limits $U_\infty\leq LU_\infty$. Again since $L(L^kU_0)\leq U_\infty$, $\sum_{i\in S}(b_i+\alpha L^kU_0(V_i))\leq U_\infty(V)$ for any $(b_i, V_i)_{i\in S}\in\Lambda(V)$. So taking limits $\sum_{i\in S}\pi_i(-b_i+\alpha U_\infty(V_i))\leq U_\infty(V)$ and taking the supremum, $LU_\infty(V)\leq U_\infty(V)$.

LEMMA 4. *Assuming $U_{k-1}(V)$ is strictly concave, at the solution to (2): (i) the local downward incentive compatibilty constraints always bind, (ii) there is coinsurance, i.e. $-b_i+\alpha U(V_i)\geq -b_{i-1}+\alpha U(V_{i-1})$ and $v(b_i+\theta_i)+\alpha V_i>v(b_{i-1}+\theta_{i-1})+\alpha V_{i-1}$, (iii) the local upward incentive compatibility constraints never bind.*

Proof. (i) It is first shown that $C_{i,i-1}=0$. Suppose not. Then $V_i>V_{i-1}$ for some i and consider changing $(b_i, V_i)_{i\in S}$,as follows: keep V_1 fixed and reduce V_2 until $C_{2,1}=0$. Next reduce V_3 until $C_{3,2}=0$, and so on, until $C_{i,i-1}=0$ for all i. Add the necessary constant to each V_i to leave EV_i unchanged overall. Each (V_i-V_{i-1}) has been reduced so the lender's utility is increased. Moreover the new contract offers the borrower the same utility and is incentive compatible since $b_{i-1}\geq b_i$ for all i (these are unchanged) and together with the binding downward constraints this implies the upward constraints hold. Hence the original contract has been improved, contrary to assumption.

(ii) The latter follows from part *(i)* . So suppose $-b_i+\alpha U(V_i)<-b_{i-1}+\alpha U(V_{i-1})$. Then replacing b_i by b_{i-1} and V_i by V_{i-1} raises the lender's utility but leaves the borrower's utility unchanged. It is also incentive compatible.

(iii) Suppose we ignore the constraint $b_{i-1}\geq b_i$ and the solution has $b_i>b_{i-1}$. Then by *(ii)* $V_i<V_{i-1}$ and replacing b_{i-1} by b_i and V_{i-1} by V_i cannot decrease the lender's utility and cannot violate incentive compatibility. But $v(b_i+\theta_{i-1})-v(b_{i-1}+\theta_{i-1})>v(b_i+\theta_i)-v(b_{i-1}+\theta_i)$ $=\alpha(V_{i-1}-V_i)$ since v is concave. So $v(b_i+\theta_{i-1})+\alpha V_i>v(b_{i-1}+\theta_{i-1})+\alpha V_{i-1}$ and the borrower's utility is also improved.

Proof of Proposition 1. It is obvious that $U_k(V)$ is decreasing and if it is concave Lemma 1 of [2] shows that it is continuously differentiable. Assume $U_{k-1}(V)$ is strictly concave. Consider any V and V' with the associated contracts $(b_i, V_i)_{i\in S}$, $(b_i', V_i')_{i\in S}$. Let $V_i^*=\delta V_i+(1-\delta)V_i'$ and define b_i^* by $v(b_i^*+\theta_i)=\delta v(b_i+\theta_i)+(1-\delta)v(b_i'+\theta_i)$, for $\delta\in(0, 1)$. Then $C_{i,}{}^*{}_{i-1}=\delta C_{i,i-1}+(1-\delta)C_{i,}{}'{}_{i-1}+\delta v(b_{i-1}+\theta_i)+(1-\delta)v(b_{i-1}'+\theta_i)-v(b_{i-1}^*+\theta_i)$. By Lemma 4, at the optimum, $C_{i,i-1}=0$ and $C_{i,}{}'{}_{i-1}=0$ and since the risk premium is a decreasing function of income (Assumption 1) the third term is non-negative. However the contract $(b_i^*, V_i^*)_{i\in S}$ may violate the upward incentive constraints. Nevertheless using a similar argument to that used in Lemma 4 (i) it is possible to construct a new contract from $(b_i^*, V_i^*)_{i\in S}$ which is incentive compatible and offers both the lender and the borrower no less utility. Strict concavity then follows because it is not possible to have both $b_i=b_i'$ and $V_i=V_i'$ for all $i\in S$ and $V\neq V'$.

Proof of Proposition 3. The proof proceeds in three stages.

(i) U_*' is a non-positive martingale. Therefore by Doob's Convergence Theorem ([4], p.204) it converges almost surely to some random variable, R. Since $Lim_{c\to a}v'(c)=-\infty$, $Lim_{V\to-\infty}U_*'=0$ and it suffices to show $R=0$.

(ii) It is shown for any $V_t\in(-\infty, d_\infty)$ and $\gamma<U_*'(V_t)$, $Pr\{U_*'(V_{t+\tau})\leq\gamma: V_t\}\equiv\sigma>0$ for some τ. Consider a sequence in which state S is repeated τ times consecutively. Then $U_*'(V_{t+\tau})\leq U_*'(V_t)$ since $V_S>V$ (this is tedious but straightforward to prove). Suppose no τ exists such that $U'(V_{t+\tau})\leq\gamma$. Then $Lim_{\tau\to\infty}U_*'(V_t)>\gamma$ or equivalently $Lim_{\tau\to\infty}V_{t+\tau}<\phi$ where $U_*'(\phi)=\gamma$, both limits exist by the concavity of U_*. Also $Lim_{\tau\to\infty}|U_*'(V_{t+\tau})-U_*'(V_{t+\tau-1})|=0$ and using (6) each $\mu_i\to 0$. By continuity of the contract in V, for $V=Lim_{\tau\to\infty}V_{t+\tau}$, $V_S=Lim_{\tau\to\infty}V_{t+\tau}$, a contradiction. Thus τ exists and the probability of the sequence is $\pi_S{}^\tau>0$.

(iii) If $R\neq 0$ there must be a negative interval (g,h) such that $Pr\{Lim_{\tau\to\infty}U_*'(V_t)\in(g,h)\}=\xi$ for some $\xi>0$. By Egoroff's Theorem ([8], p.199) since U_*' converges almost surely, for every $\delta>0$ and $\varepsilon>0$ there exists an integer n such that $Pr\{|U_*'(V_t)-R|<\varepsilon,\ t\geq n\}>1-\delta$. Choosing ε small enough so that $(U_*')^{-1}(h+\varepsilon)$ exists and letting $\delta=\xi/2$, then $Pr\{U_*'(V_t)\in(g-\varepsilon,\ h+\varepsilon),\ t\geq n\}>\xi/2>0$. Let $Pr\{U_*'(V_n)\in(g-\varepsilon,\ h+\varepsilon)\}=\psi$. By *(ii)* there exists a σ and a τ such that a proportion σ of the paths will leave the interval within τ periods. So after $m\tau$ periods $Pr\{U_*'(V_t)\in(g-\varepsilon,\ h+\varepsilon),\ t=n,n+1,\dots,n+m\tau\}\leq(1-\sigma)^m$. So in the limit, $Pr\{U^{*\prime}(V_t)\in(g-\varepsilon,\ h+\varepsilon),\ t\geq n\}=0$, a contradiction.

Proof of Proposition5. By the mean value theorem there is some ϑ_i^α between V_i^α and V such that $h_\alpha(V_i^\alpha)=U^{\alpha\prime}(\vartheta_i^\alpha)(V_i^\alpha-V)$. From the incentive constraints $V_i^\alpha-V_{i-1}^\alpha=(1-\alpha)B_i/\alpha$ where $B_i\equiv v(b_{i-1}^*+\theta_i)-v(b_i^*+\theta_i)$ is a constant independent of α. By feasibility $\Sigma_{i\in S}\pi_iV_i^\alpha=V$ and it is simple but tedious to show

$$\alpha/(1-\alpha)\sum_{i\in S}\pi_ih_\alpha(V_i^\alpha)=\sum_{i=2}^{S}\pi_i\sum_{j<i}\pi_j\sum_{k=i}^{j+1}B_k(U^{\alpha\prime}(\vartheta_i^\alpha)-U^{\alpha\prime}(\vartheta_j^\alpha)).$$

Clearly $V_i^\alpha\rightarrow V$, and so $\vartheta_i^\alpha\rightarrow V$ for all $i\in S$. Letting U be the pointwise limit of U^α, it follows from the properties of convergent concave functions that each $U^{\alpha\prime}(\vartheta_i^\alpha)\rightarrow U'(V)$.

Proof of Proposition 6. For simplicity (and without loss of generality) assume there are just two states. By incentive compatibility in the first period $V_2^\alpha-V_1^\alpha=((1-\alpha)/\alpha)(v(b_1^\alpha+\theta_2)-v(b_2^\alpha+\theta_2))$. The latter bracket can be shown to be bounded, so $V_i^\alpha\rightarrow V$. Since $U^{\alpha\prime}(V_2^\alpha)\rightarrow U'(V)$, from (5) and (6), $v'(b_i^\alpha+\theta_i)\rightarrow -1/U'(V)=v'(b_i^*+\theta_i)$. Therefore $b_i^\alpha\rightarrow b_i^*$. Second period payments in state j, b_{ij}^α depends on V_i^α, as does V_{ij}^α. So reapplying the same arguments, $V_{ij}^\alpha\rightarrow V$, $b_{ij}^\alpha\rightarrow b_j^*$ and so on.

Proof of Proposition 7. The solution is clearly feasible. It is also easy to check that U_* is a fixed point of L. The other conditions are derived directly from (5), (6) and(7).

Proof of Proposition 8. We take each part of the proof in turn.

(i) Since $a_i\geq a_j$, and $d_i\leq d_j$ for $i<j$, $r_{ij}\geq 0$. As log is concave $r_{ij}=-(logd_i-logd_{i-1})/(loga_i-loga_{i-1})<-(d_i-d_i)a_i/(a_i-a_{i-1})d_i=c_ia_i/\alpha d_i$. But from (5) and (6) $c_ia_i\leq(1-\alpha)d_i$. Therefore $r_{i,i-1}<(1-\alpha)/\alpha$. By definition $r_{i,i-2}$ is a convex combination of $r_{i,i-1}$ and $r_{i-1,i-2}$, so $r_{i,i-2}<(1-\alpha)/\alpha$. Since this is true for $i=2,3,\ldots,S$, $r_{ij}<(1-\alpha)/\alpha$ for all $i,j\in S$.

(ii) By definition expected payments change each period by $-R^{-1}\Sigma_{i\in S}\pi_ilogd_i=R^{-1}\Sigma_{i\in S}\pi_ilogd_i^{-1}<R^{-1}log\Sigma_{i\in S}\pi_id_i^{-1}=0$.

(iii) If t is large τ_j can be approximated by $t\pi_j$ with probability one, so using the inequality in *(ii)* proves the result.

Footnotes

1. Allen [1] shows that if the borrower can borrow and save at the rate of time preference unobserved by the lender there is no feasible contract which shares risk.

2. The Revelation Principle applies for T finite or infinite and any stochastic structure.

3. This is a standard procedure. See for example Hart [6].

4. For greater detail in a more general model the reader is referred to Lockwood and Thomas [7].

References

1. F. ALLEN, Repeated principal-agent relationships with lending and borrowing, *Economic Letters* 17 (1985), 27-31.

2. L.M. BENVENISTE AND J.A. SCHEINKMAN, On the differentiability of the value function in dynamic models of economics, *Econometrica* 41 (1979), 727-733.

3. D. FUDENBERG, B. HOLSTROM AND P. MILGROM, Repeated moral hazard with borrowing and savings, Mimeo, Yale School of Organisation and Management, 1986.

4. G. GRIMMETT AND D. STIRZAKER, "Probability and Random Processes", Clarendon Press, Oxford, 1983.

5. S. GROSSMAN, D. LEVHARI AND L.J. MIRMAN, Consumption under uncertainty, *in* "General Equilibrium, Growth and Trade" (J. Green and J.A. Scheinkman, eds.), 105-124, Academic Press, London, 1979.

6. O.D.HART, Optimal labour contracts under asymetric information: an introduction, *Rev. Econ. Stud.* 50 (1983), 3-25.

7. B. LOCKWOOD AND J. THOMAS, Asymptotic efficiency in hidden information models, Mimeo, Warwick University, 1986.

8. M.E.MUNROE, "Measure and Integration", Addison-Wesley, Reading, Mass. (2nd edition), 1971.

9. R. RADNER, Repeated principal-agent games with discounting *Econometrica* 53, 1173-1198.

10. R. TOWNSEND, Optimal multiperiod contracts, and the gain from enduring relationships under private information, *J. Pol. Econ.* 90 (1982), 1166-1186.

11. M.E. YAARI, A law of large numbers in the theory of consumer's choice under uncertainty, *J. Econ. Theory* 12 (1976), 202-217.

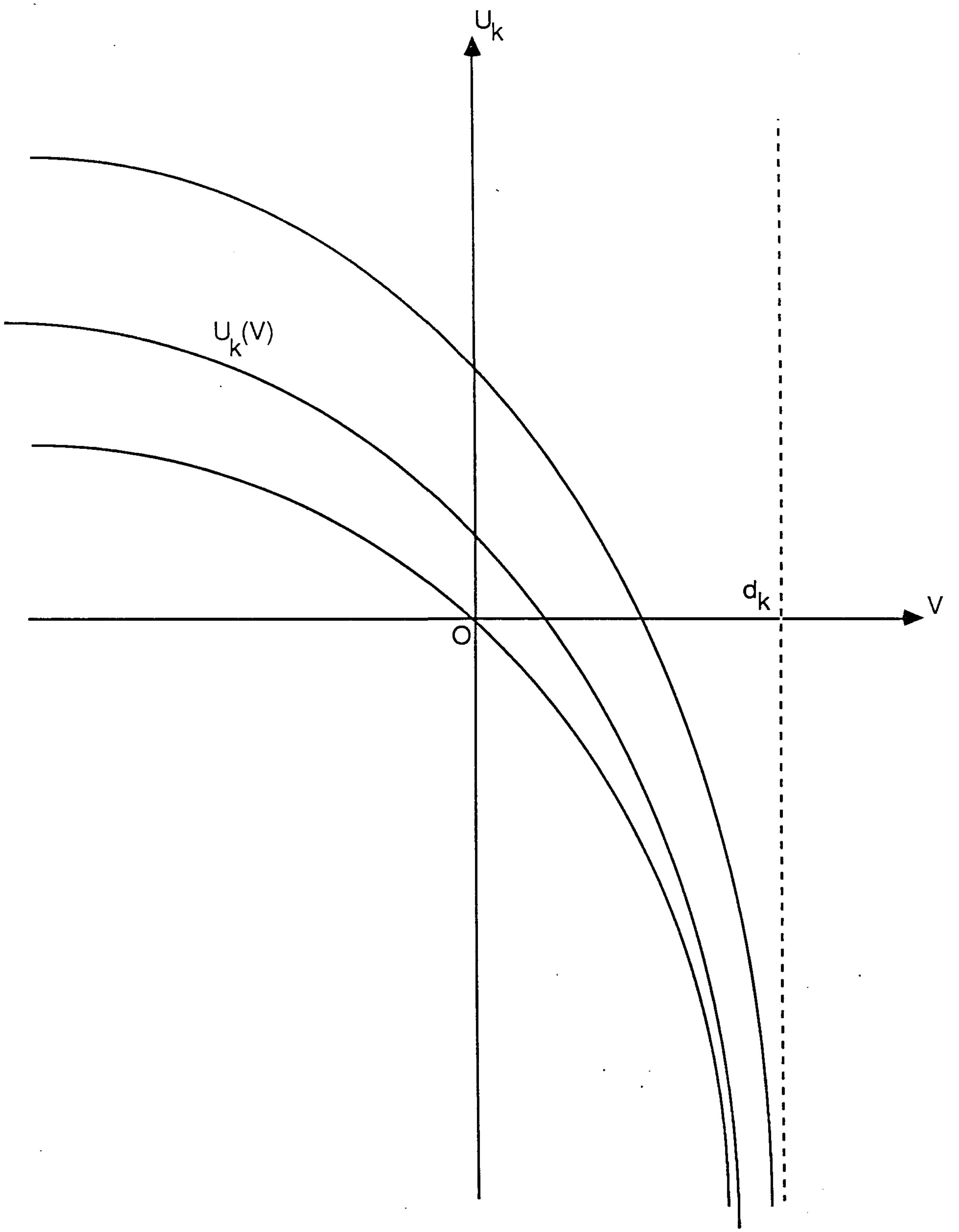

Figure 1 - The Bounds on the Value Function