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# Implicit Contracts and Asymmetric Information *

by

T. S. WORRALL

(University of Liverpool)



Department of Economic & Business Studies,
University of Liverpool,
P.O. Box 147,
Liverpool L69 3BX

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# IMPLICIT CONTRACTS AND ASYMMETRIC INFORMATION*

## Section 0: Introduction

One of the main reasons for studying implicit contracts is to examine any information asymmetry between the contracting parties. In particular it is a widely held belief that moral hazard problems are prevelant in the labour market. Azariadis for example writes in the introduction of his classic 1973 article:

> "....the obvious fact bears repeating that no market exists for direct exchange of claims on future labour services: the costs of monitoring and enforcement and 'moral hazard' are some of the reasons why such markets have not arisen."

Moral hazard can in general be defined as any situation in which the very fact that an agent is 'insured' against some outcome actually changes his best course of 'action'. Of course if the insurer can observe the agent's action then he can stipulate what the insurees action should be. Then if the insuree deviates - all bets are off. However if the insurer cannot observe the insurees action, there is an information asymmetry, then the insurance coverage will have to be reduced, possibly to zero![1]

Moral hazard can arise on both sides of the labour market. For example it is difficult for an employee to sell his labour services contingent upon his own disposition because this is not easily observable by the employer or indeed any other agent. This is not to say that private institutions will never provide sickness or unemployment insurance rather such insurance will never be complete that is enough to maintain a constant marginal utility of income. Anyone who has tried to borrow against their future labour income will know this to be true. On the opposite side of the labour market an employer will usually be unable to purchase labour services contingent upon say, profits unless these are independently audited.

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Consider a contract $\delta$ between an employer and an employees which specifies the labour input, $\ell$ and the employees remuneration, r in each possible state of nature $\theta$, ie. $\delta = \{r(\theta).\ell(\theta)\}$. Although Azariadis stressed the problem of moral hazard he analysed a contract where both the employer and employee could observe $\theta$ and thus where there was no information asymmetry.[2] In fact there is little point in studying such a contract since if both employer and employee can observe $\theta$ contingent markets can operate. Therefore the purpose of this chapter is to examine the optimal labour contract where there is asymmetric information, that is a contract in which one party cannot observe $\theta$. It is only recently that Grossman and Hart (1981), Green and Kahn (1982) and Hart (1983) have developed a rigorous analysis of the moral hazard problem in the labour contract context. Their work builds upon the work of Spence and Zeckhauser (1971). Harris and Raviv (1979) and Laffont and Maskin (1980) on incentives.

This paper considers two polar cases of information asymmetry. In the first case the employee observes the true state of nature but the employer cannot. Such a case is consistent with the state of nature being indentified with the employees disposition, or indeed any factor which affects the employees productivity which the employer cannot observe. In the second case the roles are reversed so that it is the employer who has the superior information. For example the employer may have better information about randomness in the production process.[3]

Section 1 presents a homely example of the first type of information asymmetry. The example concerns the relationship between a Ph.D student and his supervisor. This provides an introduction to section 2 and emphasis is placed on intuition rather than rigour.Section 2 is rather more technical, it examines the case where the employer has superior information.It is shown how this information asymmetry produces too much employment in a sense to be defined below. Section 3 concludes this chapter.

Before proceeding it is perhaps worthwhile to make a few caveats at this point. These should be remembered when reading sections 1 and 2. Firstly, Arrow has emphasized that in markets affected by moral hazard some forms of moral behaviour are likely to evolve to compensate. For example the relationship between the employer and the employee might be governed by established codes of conduct or ethical standards different from those predicted by economic or optimizing behaviour. Secondly and not unrelatedly is the issue of reputation. Once time is explicitly introduced into the analysis an agents concern for his reputation becomes important, and the temporal nature of information flows must be taken into account. These are thorny problems and no resolution is readily available, though they are obviously important to an understanding of labour contracts since there are predicated upon a stability in the employer/employee relationship.

Section 1 : How to pay for a doctoral thesis

This section asks the question: What is the optimal grant award scheme for a doctoral student? It is assumed that the students research is financed out of his supervisors salary. If the student is better informed about how he conducts his research than his supervisor then choosing the optimal payment scheme is equivalent to finding the optimal contract between the student and his supervisor under asymmetric information. The analysis can be easily reinterpreted in the labour market context by reading employee for student and employer for supervisor. It will be shown that the optimal grant to the student will be increasing and convex function of his output.

Consider a student who is enroled at the Moravian state university for a course leading to the degree of doctor of philosophy in the department of economics. The course lasts some fixed period of time, say three years. For tuition the student is assigned to some eminent and august professor who will act as a supervisor. The requirements of the course are to produce a short thesis on a particular topic, or perhaps some articles of inclusion in the learned journals of the profession.

In order to drastically simplify the problem it will be assumed that the students output y is related to his input $\ell$ in the following way

A.3.1 $\quad y = f(\ell,\theta) : [0,1] \times (a,b) \to R_+$ ; is $C^2$ and bounded above

where $\quad f_1 > 0 \quad f_2 > 0,\ f_{11} \geq 0,\ f_{12} \geq 0 \quad \lim_{\ell \to 0} f_1 = \infty$

where $\theta$ is a random variable defined on (a,b) which has a continuous probability density function $g(\theta)$. Some comments are obviously called for about the assumption A.3.1. First notice that y is a undimensional variable so that the quality and quantity of the students output are commensurable, and that y can be objectively assessed and is readily apparent to both the student and the supervisor. Second the students input or effort or work load again is undimensional. This is not however the only input. The professor will also provide an input through his supervision and encouragement of the student. For simplicity it is assumed that the professors input is constant and it is therefore supressed in the notation. Third the random variable $\theta$ is to be interpreted as a shock to the efficiency of the student since a high value of $\theta$ is associated with a high value of the students marginal product. Of course in general $\theta$ will be a vector of random variables the components of which might be luck, the skill or lack of skill of the student, the degree of difficulty or simplicity of the subject matter, the students personal circumstances etc.

However again for simplicity it will be assumed that $\theta$ is a single parameter and not a vector of parameters. What is more it is assumed that the student chooses his labour input after the random variable $\theta$ has been drawn from $g(\theta)$ and that the professor cannot observe this drawing. To motivate these assumptions suppose that $\theta$ represents the easiness of the subject area chosen by the student. That is to say a high value of $\theta$ indicates that the subject area is very easy and a low value of $\theta$ indicates that the subject is particularly difficult. Of course to actually ascertain the true value of $\theta$ requires a considerable amount of effort and investment of resources. For the first year of the degree course the student is devoted almost solely to this task. He will then decide on how much work or effort to supply in the following two years dependent upon the actual degree of difficulty of the subject. The professor however is far too busy with his own work or with other students to find out the true value of $\theta$. Of course if the professor can observe $\ell$ and knows f then he can deduce $\theta$ by inversion so it will be assumed quite reasonably that the professor does not observe the students input, $\ell$ .

The professor has to choose how much of his salary, X he is to give to the student as a grant to fund his research. The grant will be denoted r. Since the professor cannot observe either $\theta$ or $\ell$ , he will make the grant r conditional upon the students output y. This is shown rigorously later on. For the moment notice that the grant r may be properly interpreted as the amount of patronage. This of course was a common form of funding scientific work in previous centuries.

The professor will however not give anything for nothing. He will not patronize the student unless this increases his utility in some appropriately defined sense. There are at least two ways in which the students output may affect the professors utility. First the students efforts may stimulate or encourage the professor in his own work. Second the professor may derive a certain kudos from his patronage or his students work. It will be assumed that the professors utility v has the following simple functional form.

A.3.2 $\quad v = v(I) : R_+ \to R : v(I) = I = X + y - r$

where I is the professors gross income. Thus it is assumed that the students output y is commensurable with the professors net income X-r, and that the professor is risk neutral. There is really no justification for the former assumption. However if the professor is thought of as an employer then I represents profits. The professor's risk neturality might be justified if, for example, he supervised a large number of students for each of whom the drawing of $\theta$ is independent of any other students drawing.

The students utility u, depends both on the grant r and effort $\ell$ . The utility function u is assumed to satisfy

A.3.3 $\quad u = u(r,\ell) : R_+ \times [0,1] \quad R \; ; \; C^2$ and

$$u_1 > 0,\ u_2 > 0,\ u_{11} \le 0,\ u_{22} \le 0,\ s = -u_2/u_1 \ge 0$$

$$(u_{11}s + u_{12}) \le 0 (u_{12}s + u_{22}) \le 0 \ (u_{11}u_{22} - u_{12}^2) \ge 0.$$

Before proceeding it is convenient to facilitate the graphical and algebraic analysis by rewriting the production and utility functions. The production $y = f(\ell,\theta)$ can be inverted to give

$$\ell = h(y,\theta) h_1 = 1/f_1 > 0,\ h_2 = -f_2/f_1 < 0,\ h_{11} = -f_{11}/f_1^3 \ge 0,$$

$$h_{12} = (-1/f_1^3)(f_1 f_{12} - f_{11} f_2) \le 0. \tag{3.1}$$

Using equation (3.1) the students utility function can be rewritten as

$$u = u(r,y,\theta) = u(r,h(y,\theta)) \tag{3.2}$$

which can also be inverted to give

$$r = r(u, h(y, \theta)) \; ; \quad r_1 = 1/u_1 > 0 \; r_2 = s > 0 \qquad (3.3)$$

It is now possible to examine exactly what is meant by a contract with asymmetric information. In chapter 1A it was assumed that the components r and $\ell$ of a contract could be made contingent upon the state of nature $\theta$. However it is less clear how such a contract can be impmemented if there is asymmetric information. In particular it might be assumed that the student will have an incentive to lie about the true state of nature. Nevertheless Dasgupta, Hammond and Maskin (1979) have shown that any contract that can be implemented is implementable by a contract in which there is no incentive to lie. This has been demonstrated graphically by Harris and Townsend (1981).

Harris and Townsend's analysis can be adapted to the present case. First consider the students and the professors indifference maps in (r,y) space. These are drawn in diagrams 3.1 and 3.2. The professor's indifference curves are $45^\circ$ lines and his utility increases towards the south-east. The students indifference curves are convex to the south-east and his utility increases towards the north-west. However the student has a different indifference map for every state of nature. Suppose for the moment that there are just two states of nature $\theta_1$ and $\theta_2$. That is to say that the students chosen subject is hard or harder still. Then at any given point in (r,y) space the students indifference curve associated with $\theta = \theta_2$ has a lower slope than the indifference curve associated with $\theta = \theta_1$.

Consider the fixed income contract $(c_1, c_2)$ depicted in diagram 3.3 where $c_1 = (r(\theta_1), y(\theta_1))$ and $c_2 = (r(\theta_2), y(\theta_2))$. Clearly the student prefers the allocation $c_1$ to $c_2$ which ever state occurs. If the state of nature is $\theta = \theta_2$ then the student gets higher utility at the point $c_1$. Similalry if $\theta = \theta_1$ the students utility is lower utility at the allocation $c_2$.

Thus if the supervisor cannot observe $\theta$ the student will announce that the state of nature is $\theta = \theta_1$ regardless of the true state of nature and therefore the contract $(c_1, c_2)$ is not implementable. This is quite natural, since the student receives the same remuneration independent of his output he will always choose to produce less output because this requires less effort. Notice that the fixed income contract $(c_1, c_2)$ is typical of the sort of contract offered to doctoral students in the U.K. Therefore it seems quite likely that U.K. students will always claim that their subject is especially difficult and produce low quality output.[5]

On the other hand consider the alternative contract $(c_1, c_2')$ where $c_1 = (r(\theta_1), y(\theta_1))$ and $c_2' = (r'(\theta_2), y'(\theta_2))$. This contract is incentive compatible. That is the student will never have any incentive to lie about the true state of nature. For instance if $\theta_2$ is the true state of nature the student cannot gain by announcing the true state of nature is $\theta_1$ and if $\theta_1$ is the true state of nature he will always benefit by announcing the true state to be $\theta_2$. Thus it can be seen that the only contracts that are implementable are incentive compatible and any incentive compatible contract can be implemented by the student telling the truth.

Incentive comptaible contracts have some very interesting properties. Examination of all of these properties must await a complete algebraic treatment but diagram 3.4 indicates that not all first best contracts are incentive compatible. In chapter 1A it was shown that the optimal contract equated the marginal disutility of labour to the marginal product of labour and left the employees marginal utility of income independent of the state of nature. Such a contract is called a first best contract because it is pareto efficient. An example of a first best contract is $(c_1, c_2'')$ drawn in Diagram 3.4. The dotted lines

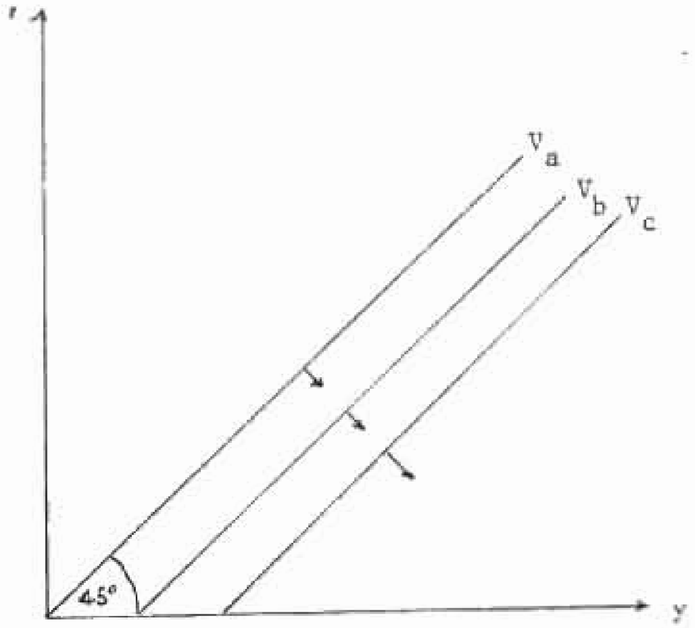


Diagram 3.1

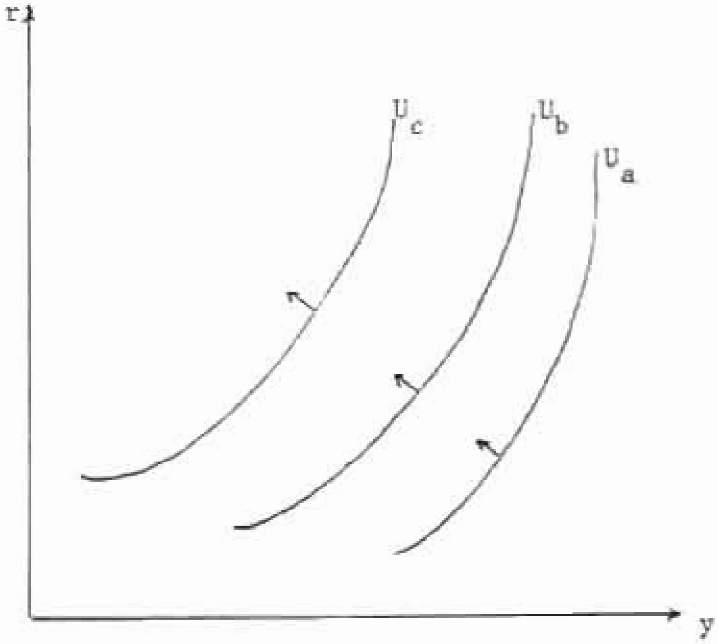


Diagram 3.2

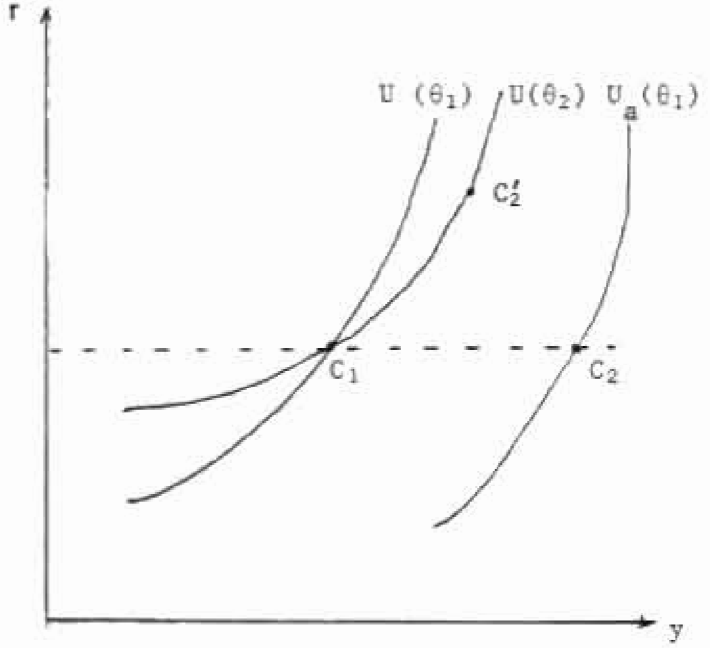


Diagram 3.3

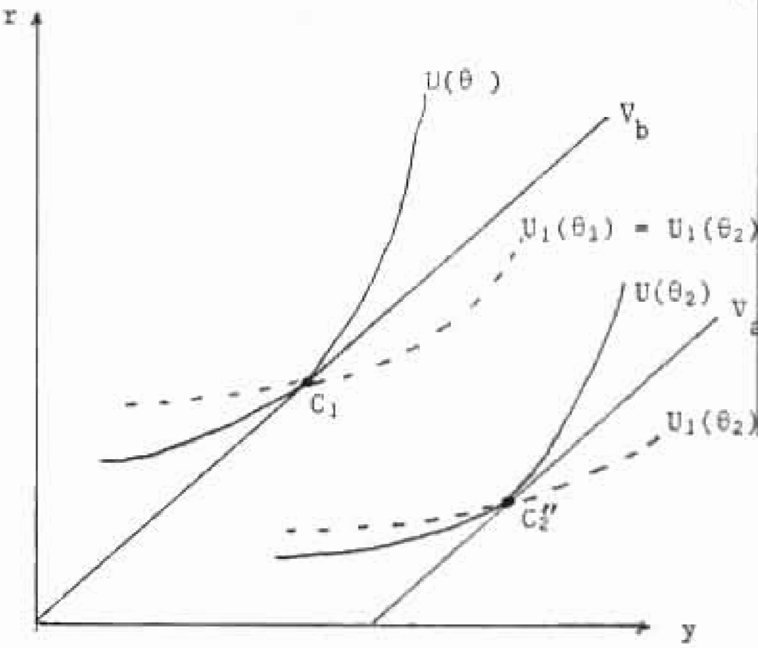


Diagram 3.4

represent lines of equal marginal utility of income. Since labour is a normal good they have a lower slope than indifference curves. The two dotted lines through $c_1$ and $c_2''$ represent the same level of marginal utility but in different states. However the contract $(c_1, c_2'')$ is not incentive compatible because the student will always announce the state to be $\theta = \theta_1$ independent of the true state of nature. In fact it can be shown that the first best contract is only incentive compatible if labour is neither a normal nor inferior good. This was the basis for most of the analysis in chapter 1.

The algebraic analysis is most easily understood if it is assumed that there is a continuum of possible states of nature so that the density function $g(\theta)$ is continuous. Now it has been shown that an incentive compatible contract can be written $\delta = \{r(\theta'), y(\theta')\}$ in precisely the same way that the first best contract could except is that value of $\theta$ the student chooses to announce. Given the contract $\delta$ the student is in a position to announce any state of nature he perceives to be to his advantage. That is the student will choose to announce that state of nature $\theta'$ that maximizes his utility when the true state of nature is $\theta$ . Therefore for each possible value of $\theta$ the student sovles :

P.3.1 $$\max_{\theta'} \quad u(r(\theta'),\ h(y(\theta'),\theta)) \quad \text{s.t } \theta' \in [a,b]$$

A contract $\delta$ is incentive compatible if the student always announces the true state of nature, or equivalent if the solution to P.3.1 is $\theta'(\theta) = \theta$. The first and second order conditions for P.3.1 are

$$\dot{r}(\theta) = s(r(\theta),\ h(y(\theta),\theta))\ h_1(y(\theta)\ ,\theta)\ \dot{y}(\theta) \tag{3.4}$$

$$\ddot{r}\ - sh_1\ddot{y} - \dot{y}^2\ (sh_{11} + h_1^2(s\ s_1 + s_2)) \leq 0 \tag{3.5}$$

Totally differentiating equation (3.4) equation (3.5) can more conveniently be written as

$$\dot{y}(sh_{12} + s_2h_1h_2) \geq 0 \tag{3.6}$$

Given assumptions A.1 and A.3 equation (3.6) implies that $y(\theta)$ is positive. It will be convenient to take $y(\theta)$ to be strictly positive in what follows though it will be shown in section 2 that the results are unaffected. Then inverting $y = y(\theta)$

$$\theta = y^{-1}(y) \qquad (3.7$$

and

$$r(\theta) = r(y^{-1}(y)) = r(y) \tag{3.8}$$

Equation (3.8) shows that remuneration is dependent upon output. It also shows that the contract $\delta$ can be viewed alternatively as choosing the reward function $r(y)$. This is essentially a piece-rate system of operation. The student chooses how much output to produce given the reward scheme $r(y)$. That is in each state of nature the student solves

P.3.1′ $\quad \max\limits_{y} \quad u(r(y),\ h(y,\ .)) \quad s.t \quad h(y,\ .)\ \epsilon[0,1]$

The first and second order conditions for P.3.1 are

$$r'(y) = s(r(y),\ h(y,.)\ h\ (y,.) \tag{3.4′}$$

$$r''(y) \leq sh_{11} + h_1^2(s_1s + s_2) \tag{3.5′}$$

Differentiating equation (3.8) shows that equations (3.4′) and (3.5′) are the same as equations (3.4) and (3.5) and therefore P.3.1 and P.3.1′ are equivalent. Equation (3.4′) has a natural interpretation. It states that the student will equate the marginal rate of substitution between income and leisure to the marginal benefit of supplying labour which is

$$\partial r/\partial \ell = r'(y)/h_1(y,.) \tag{3.9}$$

Notice that the student can only do this because it is being assumed that the professor cannot observe the students labour input. Consequently in order to induce the student to produce good quality work the reward offered must increase as output increases.

Perhaps the most natural way to write the incentive compatible contract is $\delta = \{u(\theta), y(\theta)\}$ . The output schedule $y(\theta)$ is the solution to P.3.1 and is chosen directly by the student. The schedule $u(\theta)$ is the maximum value function for P.3.1 . Once $u(\theta)$ and $u(\theta)$ are determined $\ell(\theta)$, $r(\theta)$ and $v(\theta)$ are determined directly by equation (3.1) and (3.3) and assumption A.3.2.

The contract $\delta$ must offer the student a given level of expected utility $\bar{u}$ or else the student will enrole at another university or perhaps take up a non-academic post, therefore

$$\int_a^b u(\theta)g(\theta)d\theta = \bar{u} \tag{3.10}$$

It must also satisfy the incentive compatibility constraint (3.4′). This can be rewritten in terms of $y(\theta)$ and $u(\theta)$ as

$$\dot{u}(\theta) = u_2(r(u(\theta), h(y(\theta),\theta)), h(y(\theta),\theta)).h_2(y(\theta),\theta) \tag{3.4''}$$

It will also be assumed that the set of contracts satisfying equations (3.10) and (3.4″) $\Delta$ is non-empty.[6] The professor's expected utility is

$$\int_a^b (X + y(\theta) - r(u(\theta), h(y(\theta),\theta)))\ g(\theta)d\theta \tag{3.11}$$

The optimal contract $\delta^*$ maximizes the professors utility (3.11) subject to the constraints (3.10) and (3.4″). This is an optimal control problem, where $y(\theta)$ is the control variable, controlled by the student and $u(\theta)$ is the state variable. Letting $\lambda$ be the multiplier for equation (3.10) and $p(\theta)$ be the costate variable for equation (3.4″) for the Hamiltonain equation is

$$H(y(\theta),\ u(\theta), \lambda\ ,\ p(\theta)) = (y(\theta) - r(u(\theta),\ h(y(\theta),\theta)) + \lambda\ u(\theta))g(\theta)$$

$$+ p(\theta)\ u_2(r(u(\theta),\ h(y(\theta)\ ,\theta)),\ h(y(\theta),\theta)).h_2(y(\theta),\theta). \tag{3.12}$$

The optimal contract $\delta^*$ is the solution to

P.3.2 $\qquad \max\limits_{\delta} \qquad H(\delta, \lambda\ p(\theta))$ .

The important point to grasp is the sign of the costate variable. From equation (3.4″) it is clear that $u(\theta)$ is positive. However if the professor could observe $\theta$ then the optimal contract is the first best contract in which case $u(\theta)$ is negative. Therefore the incentive compatibility constraint contrains $u(\theta)$ from below and therefore $p(\theta)$ is negative.

Differentiating the Hamiltonian with respect to the control variable $y(\theta)$ gives directly

$$(1 - r_2h_1) = pu_1(sh_{12} + s_2h_1h_2)/g > 0. \tag{3.13}$$

Equation (3.13) states that the marginal rate of substitution between income and leisure $r_2 = s$ is less than the marginal product of labour $f_1 = 1/h$. That is to say in any particular state of nature, or ex post both the professor and student could be made better of by some increase in remuneration or effort. To put it more strikingly there is underemployment. This cannot however be described as involuntary unemployment since the contract is ex ante optimal, that is there is no way in which to increase both the professors and the students expected utility. In this sense the contract is constrained pareto efficient. It is also clear that the student will not wish to supply any more labour at the going wage because the wage or the marginal benefit to supplying labour is equated to the marginal rate of substitution.

On the other hand the professor will always prefer that labour supply is greater at the going wage but it is less clear how important this is because the wage is not taken parametrically but varies as the students effort changes.

Some insight into this result can be obtained by comparing the first best contract with the optimal incentive compatible contract. The first best contract equates the marginal rate of substitution between income and leisure to the marginal product but both exceed the marginal benefit to supplying labour. That is

$$\partial r/\partial \ell \leq s(r,\ell) = f_1(\ell,\theta) \qquad \forall\, \theta \tag{3.14}$$

whereas the optimal incentive compatible contract

$$\partial r/\partial \ell = s(r,\ell) \leq f_1(\ell,\theta) \qquad \forall\, \theta \tag{3.15}$$

so that at least the divergence between the marginal cost of labour and the marginal product is preserved.

To conclude, since the professor cannot observe the students input he is quite likely to under supply and produce low quality output. These effects are amelioated by making the students grant an increasing function of output. It is tempting to suggest that if welfare is assessed purely on an ex post basis then an increase in grant and effort would be beneficial to all parties.

Section 2 : More on Incentive Compatible Contracts

In this section the employer/employee model is rehabilitated but is now assumed that the employer can verify which state of nature has occurred but that the employee cannot. It is not difficult to think of examples where this might be the case. For example the firm may have more information about the demand conditions affecting the firm or more information about the operation and effectiveness of other inputs into the production process. This section examines the optimal incentive compatible contract in this situation.

Assumption A.3.1 - A.3.3 of section 1 are maintained throughout this section but the information sets of the employer and employee now satisfy

A.3.4
$$I^v = \{\theta, y, r, \pi, \ell, u, v, f, g\}$$
$$I^u \quad \{r, \ell, u, v, f, g\}$$

where $I^v$ is the employers information set and $I^u$ is the employees information set. Notice that there is really no contradiction in the employee knowing the objective probability density function $g(\theta)$ and not the realized value of $\theta$. For example the employee may be able to learn the true value of $\theta$ with a one period lag, then after some time he will be able to deduce $g(\theta)$ but still as far as the present contract is concerned be unable to observe the current $\theta$. Alternatively with the important exception of lemma 3 the analysis goes through if the employee has a subjective probability density function $k(\theta)$. The coincidence of $g(\theta)$ and $k(\theta)$ can be seen as a strong rational expectations assumption. Equally the fact that f is included in $I^u$ is not restrictive since any uncertainty about f might be included with the state of nature $\theta$. But this explains why y and $\pi$ are not included in $I^u$ since observing either output or profits is equivalent to knowledge of $\theta$ through the knowledge of f and v. This being said the examination of the optimal incentive can proceed along very similar lines to the analysis of section 1 except that more attention will be paid to specific details.

In section 1 it was shown that any contract under asymmetric information could be written as a state contingent contract. In this section the employer is assumed to be able to observe $\theta$ but the employee cannot. Therefore the employer will choose to announce that state of nature $\theta'$ that is in his own best interest. In general the employers profits will depend both upon the true state of nature $\theta$ and the state of nature $\theta'$ the employer chooses to announce has occurred. The employer will choose to announce $\theta'$ so as to maximize his ex post utility or profits, that is to say he solves

$$\text{P.3.3} \qquad \max_{\theta'} \quad \pi(\theta',\theta) \quad \text{s.t} \quad \theta' \varepsilon [a,b]$$

The solution set for P.3.3 is $\theta'(\theta)$ where

$$\theta'(\theta) = \{\theta' \varepsilon [a\ b] \quad \pi(\theta',\theta) \geq \pi(\theta'',\theta) \ \forall \ \theta'' \ \varepsilon [a,b]\} \tag{3.16}$$

A contract $\delta$ is said to be incentive compatible if and only if the employer never has any incentive to lie, that is

$$\theta \ \varepsilon \ \theta'(\theta) \tag{3.17}$$

There are two points to be made about equation (3.17). First it is being assumed that the employer will report the state of nature honestly unless he can actually gain by lying. This is equivalent to saying that if the employer is indifferent between any two allocations he will always choose the one that the employee most prefers. Second incentive compatibility implies that the solution set to P.3.3 $\theta'(\theta)$ is non-empty. This is guaranteed if $\pi(\theta',\theta)$ is upper semi-continuous in the choice of variable $\theta'$.

The maximum value function for P.3.3. is

$$\pi(\theta) = \pi(\theta,\theta) \tag{3.18}$$

which is continuous if $\pi(\theta',\theta)$ is continuous in $\theta'$ and $\theta$. In fact the absolute continuity of $\pi(\theta)$ will be assumed below but for the moment suppose that $\pi(\theta',\theta)$ is continuous and twice differentiable. Then the first and second order condition for P.3.3 can be written as

$$\pi_{\theta'}(\theta',\theta) = 0 \tag{3.19}$$

$$\pi_{\theta'\theta'}(\theta',\theta) \leq 0 \tag{3.20}$$

If equation (3.20) is satisfied as an inequality the solutions of $\theta'(\theta)$ will be a continuous and differentiable function. Then the following four equations are all equivalent definitions of incentive compatibility

$$\theta'(\theta) = \theta \tag{3.21}$$

$$\partial\theta'(\theta)/\partial\theta = 1 \qquad (3.21)$$

$$\pi_{\theta'\theta'}(\theta'(\theta),\theta) + \pi_{\theta'\theta}(\theta'(\theta),\theta) = 0 \tag{3.23}$$

$$\pi(\theta) = \pi_{\theta'}(\theta'(\theta),\theta) + \pi_{\theta}(\theta'(\theta),\theta) = \pi_{\theta}(\theta,\theta) \tag{3.24}$$

Equation (3.22) is obtained by differentiating equation (3.21). Equation (3.23) is obtained by totally differentiating equation (3.19) and using equation (3.22). Equation (3.24) is obtained by differentiating equation (3.18) and using equation (3.19). Given equation (3.23) the second order condition equation (3.20) can be conveniently rewritten

$$\pi_{\theta'\theta}(\theta'(\theta),\theta) \geq 0 \quad . \tag{3.25}$$

It is perhaps more helpful to write the profit function $\pi(\theta',\theta)$ explicitly as revenue minus costs. That is

$$\pi(\theta',\theta) = f(\ell(\theta'),\theta) - r(\theta') \tag{3.26}$$

where $r$ is remuneration and $\ell$ is labour input. Therefore rewriting equation (3.24) incentive compatibility implies

$$\dot{\pi}(\theta) = f_2(\ell(\theta),\theta) \ . \tag{3.27}$$

This will be referred to as the incentive compatibility constraint. It constrains profits to increase with $\theta$ only to the extent that output increases with $\theta$ . Again rewriting equation (3.25)

$$\pi_{\theta'\theta}\ (\theta'(\theta),\theta)\big|_{\theta'=\theta} = f_{12}(\ell(\theta),\theta)\ \dot{\ell}(\theta) \geq 0 \tag{3.28}$$

This suggests yet another interpretation of incentive compatibility. Suppose that $\dot{\ell}(\theta)$ is strictly positive then the function $\ell = \ell(\theta)$ can be inverted so that $\theta = \ell^{-1}(\theta)$ and $r(\theta) = r(\ell^{-1}(\ell)) = r(\ell)$. Thus remuneration is some function of labour input, there is essentially a time-rate system of payments. Choosing a contract is equivalent to choosing the optimal time-rate system. This does not seem inconsistent with modern industrial experience. In particular wages usually rise with hours worked, and the latter is often chosen unilaterally by the employer. In this sense incentive compatibility is reduced to the employer choosing the labour input to maximize ex post profits given the function $r = r(\ell)$. That is to say the employer solves

P.3.3′ $\qquad \max\limits_{\ell} \qquad f(\ell,\theta) - r(\ell) \qquad \text{s.t } \ell \in [0,1]$

The first and second order conditions for P.3.3 are

$$\partial r(\ell)/\partial\ell = f_1(\ell,\theta) \tag{3.29}$$

$$\partial^2 r(\ell)/\partial\ell^2 \geq f_{11}(\ell,\theta) \quad . \tag{3.30}$$

Equation (3.29) shows that the employer will equate the marginal cost of hiring labour to the marginal product of labour. Equations (3.29) and (3.30) can be rewritten as

$$\dot{r}(\theta) = f_1(\ell(\theta),\theta)\dot{\ell}(\theta) \tag{3.29′}$$

$$\ddot{r}(\theta) = f_{11}(\ell(\theta),\theta)\ddot{\ell}(\theta) + f_1(\ell(\theta),\theta)(\dot{\ell}(\theta))^2 \tag{3.30′}$$

since
$$\dot{r}(\theta) = (\partial r(\ell)/\partial\ell)\dot{\ell}(\theta) \tag{3.31}$$

and
$$\ddot{r}(\theta) = (\partial^2 r(\ell)/\partial\ell^2)\ddot{\ell}(\theta) + (\partial r(\ell)/\partial\ell)(\dot{\ell}(\theta))^2 \tag{3.32}$$

Equations (3.28′) and (3.30′) are simply equations (3.19) and (3.20) written explicitly in terms of the remuneration and labour input schedules. This shows that P.3.3 and P.3.3′ are equivalent. Thus the incentive compatible contract can be thought of in two possible ways. First the contract may be thought of as negotiating a pair of state contingent schedules such as $r(\theta)$ and $\ell(\theta)$ such that the employer always has an incentive to announce the true state of nature. Second the contract may be thought of as the time rate payment scheme $r(\ell)$ where control of the labour input is relinquished solely to the employer. The second interpretation is perhaps more natural but analytically it is simpler to deal with the first. In fact P.3.3′ suggests an obvious way to do this. The solution to P.3.3′ is $\ell = \ell(\theta)$ and the resulting profit function or maximum value function is

$$\pi(\theta) = f(\ell(\theta),\theta) - r(\ell(\theta)) \quad . \tag{3.33}$$

Given the incentive compatibility constraint equation (3.27) $\ell(\theta)$ can be treated as the control variable and $\pi(\theta)$ as the state variable in an optimal control problem. That is to say that a contract $\delta$ is defined as the set of pairs $\{\pi(\theta), \ell(\theta)\}$ .

Before proceeding to the analysis of the optimal incentive compatible contract, consider the following utility possibility set

$$U^* = \{(E_\pi,E_u) | E_\pi \geq 0 \; Eu \geq Eu(0,0), \; r(\theta) + \pi(\theta) = f(\ell(\theta),\theta), \; \dot{\pi}(\theta) = f_2(\ell(\theta),\theta) \; \forall \theta\}.$$

The set $U^*$ is the set of possible levels of expected utility that can be achieved by the employer and the employee when the employee cannot observe the true state of nature. This is to be contrasted with the utility possibility set when the employee can verify $\theta$

$$U = \{(E_\pi,E_u) | E_\pi \geq 0 \; E_u \geq Eu(0,0), r(\theta) + \pi(\theta) = f(\ell(\theta),\theta), \forall \theta\} \; .$$

Lemma 1 The set $U^*$ is a subset of $U$ and is convex.

Proof: 1. $U^* \subset U$. To show this consider points on the boundary of U. In chapter 1A it was shown that:

(i) $\dot{u}(\theta) \leq 0$ with strict inequality if labour is a normal good.

(ii) $\pi(\theta) = (-\dot{u}(\theta)/u(\theta)) + f_2(\theta) \geq f_2(\theta)$ with strict inequality if labour is a normal good. Therefore $U^*$ will include the boundary points of u only if labour is neither normal nor inferior.

2. $U^*$ is convex. To show that $U^*$ is convex it suffices to show that randomized contracts are never optimal. Consider the randomized contract $\tilde{\delta} = \{\tilde{\pi}(\theta'), \tilde{\ell}(\theta')\}$. This contract determines profit and labour input by some appropriate lottery if the employer announces that the state of nature is $\theta'$. Then the non-random contract $\delta = \{\pi(\theta), \ell(\theta)\}$ can be defined by

$$\pi(\theta') = E_{\tilde{\pi}}\, \tilde{\pi}(\theta') \qquad f(\ell(\theta'),\theta) = E_{\tilde{\ell}} f(\tilde{\ell}(\theta'),\theta) \quad \forall\, \theta \ .$$

Then the contract $\delta$ is incentive compatible if and only if $\tilde{\delta}$ is incentive compatible because they yield the same level of (expected) profits in each state. If $\tilde{\delta}$ is incentive compatible then $\theta' = \theta$ and therefore

$$E_{\tilde{\ell}}\tilde{\ell}(\theta) \geq \ell(\theta)$$

$$f(\ell(\theta),\theta) = E_{\tilde{\ell}} f(\tilde{\ell}(\theta),\theta) \leq f(E_{\tilde{\ell}}\ell(\theta),\theta)$$

by concavity and therefore using the concavity of the utility function

$$E_{\tilde{\delta}}\ u(f(\tilde{\ell}(\theta),\theta) - \tilde{\pi}(\theta), \ell(\theta)) \leq u(E_{\tilde{\ell}} f(\tilde{\ell}(\theta),\theta) - E_{\tilde{\pi}}\tilde{\pi}(\theta) \ ,$$

$$E_{\tilde{\ell}}\tilde{\ell}(\theta)) \leq u(f(\ell(\theta),\theta) - \pi(\theta),\ \ell(\theta)) \ .$$

That is to say the non-random contract $\delta$ offers the same level of (expected) profits in each state as the random contract $\tilde{\delta}$ and also no less (expected) utility. Therefore the random contract weakly (strongly) dominates the random contract and hence $U^*$ is weakly (strictly) convex.

Thus the optimal incentive compatible contract is non-random. The optimal incentive compatible contract $\delta^* = \{\pi^*(\theta), \ell^*(\theta)\}$ maximizes the employers expected profits subject to three constraints.[7] The first of these is quite familiar, namely that the employer must offer the employee his reservation price, $\bar{u}$. The second constraint is the incentive compatibility constraint equation (3.27) and the third constraint is the second order condition for P.3.3, that is $\dot{\ell}(\theta) \geq 0$. Thus the optimal contract is the solution to

P.3.4

$$\max_{\delta} \int_a^b \pi(\theta)g(\theta)d\theta$$

$$\text{s.t} \int_a^b u(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))g(\theta)d\theta = \bar{u} \tag{3.34}$$

$$\dot{\pi}(\theta) = f_2(\ell(\theta),\theta) \qquad \text{almost everywhere} \tag{3.27}$$

$$\dot{\ell}(\theta) \geq 0 \qquad \text{almost everywhere} \tag{3.35}$$

It will be convenient to ignore the constraint $\dot{\ell}(\theta) \geq 0$ initially and examine

P.3.4'

$$\max_{\delta} \int_a^b \pi(\theta)g(\theta)d\theta$$

$$\text{s.t} \int_a^b u(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))g(\theta)d\theta = \bar{u} \tag{3.34}$$

$$\dot{\pi}(\theta) = f_2(\ell(\theta),\theta) \quad \text{almost everywhere.} \tag{3.27}$$

This can be done without any loss of generality because the solution to P.3.4' will apply to those states where $\dot{\ell}(\theta) > 0$ in the solution to P.3.4. These issues are taken up again below.

There is a problem with P.3.4' as it stands since a solution may not exist. This is true even if $\bar{u}$ is chosen so that the constant contract $\bar{\delta} = \{\pi(\theta),\bar{\ell}\}$ is feasible, in addition to being incentive compatible. However if the following additional assumptions are made then it can be shown that a solution to P.3.4' exists.

A.3.5 i) $\pi(\theta)$ and $\ell(\theta)$ are absolutely continuous on $[a,b]$

ii) There exists a number $k$ s.t $\dot{\ell}(\theta) \leq k$ almost everywhere

iii) The reservation price $\bar{u}$ satisfies $\bar{u} \leq \{\sup Eu \mid Eu \in U^*\}$ .

Assumptions A.3.5(i) and (ii) are technical conditions, any control system that satisfies these conditions is called an inertial control system because it rules out a large number of discrete jumps. This would not appear to be unduly restrictive in the present context. Assumptiom A.3.4 (iii) guarantees that at least one feasible contract exists.

Lemma 2: Given A.3.1 : A.3.5 an optimal control $\delta^* = (\pi^*(\theta), \ell^*(\theta))$ for P.3.4 exists.

Proof: The functions, u, f and $f_2$ are continuous and $(\theta, \pi(\theta))$ can be restricted to a compact set. Therefore the conditions of theorem V 2.1 in Berkovitz (1974) are satisfied.

To show that the optimal contract is unique requires a further assumption

A.3.6 i) $u(r,\ell)$ is strictly concave in r and $\ell$

ii) $f(\ell,\theta)$ is strictly concave in $\ell$

iii) $f_2(\ell,\theta)$ is strictly cancave in $\ell$

Assumption A.3.6(i) and (ii) simply strengthen A.3.1 and A.3.3. Assumption A.3.6 (iii) will then be automatically satisfied if uncertainty is multiplicative that is

$$y = \phi(\theta)\ f(\ell) \qquad \phi'(\theta) > 0. \tag{3.36}$$

If $\lambda$ is the mulitplier for equation (3.34) and $p(\theta)$ is the costate variable for equation (3.27) then the Hamiltonian function for P.3.4′ is

$$H(\pi(\theta),\ell(\theta),p(\theta),\lambda) = (\pi(\theta) + \lambda u(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta)))g(\theta) + p(\theta)f_2(\ell(\theta),\theta)$$

The first order or necessary conditions for P.3.4′ are therefore

$$-\dot{p}(\theta) = (1 - \lambda u_1(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta)))g(\theta) \tag{3.37}$$

$$\lambda(u_1(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))\ f_1(\ell(\theta),\theta) + u_2(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta)))g(\theta)$$

$$= - p(\theta)f_{12}(\ell(\theta),\theta) \tag{3.38}$$

$$p(a) = p(b) = 0 \quad . \tag{3.39}$$

Lemma 3. Given the assumptions A.3.1 - A.3.6 the necessary conditions equations (3.37) - (3.39) are sufficient and the optimal contract $\delta^* = \{\pi^*(\theta), \ell^*(\theta)\}$ is unique.

Proof. Given A.3.1 - A.3.6 all the conditions of theorem 8.c.s. of Takayama (1974) are satisfied except $p(\theta) \geq 0$. Thus it is only necessary to show that $p(\theta) \geq 0$.

To do this suppose $p(\theta) \leq 0$ over some interval $(\theta', \theta'')$. Then by the continuity of the costate variable

$$p(\theta') = p(\theta'') = 0 \tag{3.40}$$

and substituting into equation (3.38)

$$s(\theta) - f_1(\theta) < 0 \qquad \theta \in (\theta', \theta'') \tag{3.41}$$

where

$$s(\theta) = -u_2(f(\ell(\theta),\theta) - \pi(\theta), \ell(\theta))/u_1(f(\ell(\theta),\theta) - \pi(\theta), \ell(\theta)) = -u_2(\theta)/u_1(\theta)$$

and $f_1(\theta) = f_1(\ell(\theta), \theta)$. Therefore

$$u_{11}(\theta)f_1(\theta) + u_{12}(\theta) < u_{11}(\theta)s(\theta) + u_{12}(\theta) \leq 0 \quad \theta \in (\theta', \theta'') \tag{3.42}$$

because labour is a normal good. Hence

$$\partial(1 - \lambda u_1(\theta))/\partial\theta = -\lambda(u_{11}(\theta)f_1(\theta) + u_{12}(\theta))\ell(\theta) > 0 \quad \theta \in (\theta', \theta'') \tag{3.43}$$

This is important because integrating equation (3.37)

$$\begin{aligned} -p(\theta'') &= \int_a^{\theta'} (1 - \lambda u_1(\theta))g(\theta)d\theta + \int_{\theta'}^{\theta''} (1 - \lambda u_1(\theta))g(\theta)d\theta \\ &= p(\theta') + \int_{\theta'}^{\theta''} (1 - \lambda u_1(\theta))g(\theta)d\theta \end{aligned} \tag{3.44}$$

and substituting equation (3.40) into this gives

$$\int_{\theta'}^{\theta''} (1 - \lambda u_1(\theta))g(\theta)d\theta = 0 \quad . \tag{3.45}$$

From equation (3.45) and equation (3.43) it can be seen that there exists a unique $\theta'''\varepsilon(\theta',\theta'')$ such that

$$1 - \lambda u\ (\theta''') = 0 \quad . \tag{3.46}$$

Then consider some $\theta\ \varepsilon(\theta',\theta''')$

$$- p\ (\theta) = - p(\theta') + \int_{\theta'}^{\theta} (1 - \lambda u_1(\hat{\theta}))g(\hat{\theta})d\hat{\theta}$$

$$= \int_{\theta'}^{\theta} (1 - \lambda u_1(\hat{\theta}))g(\hat{\theta})d\hat{\theta} < 0 \quad \forall\ \theta\ \ \varepsilon(\theta',\theta''') \tag{3.47}$$

where equation (3.47) follows via equations (3.40) and (3.43). However equation (3.47) shows that $p(\theta) > 0$ for all $\theta\ \varepsilon\ (\theta',\theta''')$ which contradicts the initial assertion that $p(\theta) < 0$. Similarly consider some $\theta\ \varepsilon\ (\theta''',\theta'')$ then

$$- p(\theta) = - p(\theta'') - \int_{\theta}^{\theta''} (1 - \lambda u_1(\hat{\theta}))g(\hat{\theta})d\hat{\theta}$$

$$= - \int_{\theta}^{\theta''} (1 - \lambda u_1(\hat{\theta}))g(\hat{\theta})d\hat{\theta} < 0 \quad \theta\ \varepsilon(\theta''',\theta'') \tag{3.48}$$

again using equations (3.40) and (3.43). Therefore taken together equations (3.47) and (3.48) imply that $p(\theta) > 0$ for all $\theta\ \varepsilon(\theta',\theta'')$ which contradicts the initial assertion that $p(\theta) < 0$ for all $\theta\ \varepsilon(\theta',\theta'')$. Since the choice of the interval $(\theta',\theta'')$ is arbitrary this completes the proof.

Lemma 3 is of some independent interest because of its similarity to the proof given in Seade (1982) that the optimal marginal rate of income taxation is positive. The following series of remarks or corollaries are meant to illustrate the sensitivity of the result.

Corollary 1 : If labour is a strictly normal good, that is $u_{11}s + u_{12} < 0$ then $p(\theta) > 0$ for all $\theta\ \varepsilon(a,b)$.

Proof: Suppose $p(\theta) \leq 0$ for all $\theta\ \varepsilon(\theta',\theta'')$. Then working through the arguments of Lemma 3, equation (3.41) becomes

$$s(\theta) - f_1(\theta) \leq 0 \qquad \theta \in (\theta', \theta'') \tag{3.41'}$$

and equation (3.42) becomes

$$u_{11}(\theta)f_1(\theta) + u_{12}(\theta) \leq u_{11}(\theta)s(\theta) + u_{12}(\theta) < 0 \qquad \theta \in (\theta', \theta'') \tag{3.42'}$$

but the remainder of the analysis is unaltered.

Corollary 1 is most important result. Since $p(\theta) > 0$ for all $\theta \in (a,b)$ it follows directly from equation (3.38) that the marginal product of labour is less than the marginal rate of substitution between income and labour. Thus the optimal incentive compatible labour contract is productively inefficient. In particular it is possible to reduce employment in any one state and increase both profits and utility in that state. Incentive compatibility however must mean that utility or profits are decreased in some other state.

This result needs some careful interpretation. Since the marginal rate of substitution exceeds the marginal rate of transformation there is overemployment. This does not however mean that the employment level for any particular state in the incentive compatible contract exceeds what it would be if the first best contract was operative. Equally it is difficult to say what this overemployment is involuntary even in an ex post sense. The employer is hiring labour exactly up to the point where the marginal cost of hiring labour services is just equal to the marginal product of these services, i.e. $f_1(\ell,.) = \partial r/\partial \ell$. For the employee however the marginal benefit to supplying labour is less than the marginal disutility of labour, $s(r,\ell) \geq \partial r/\partial \ell$. Thus the employee wishes to reduce his labour input at the going wage rate. This all sounds rather Keynsian and indeed the optimal incentive compatible contract has done away with the second classical postulate namely that $s(r,\ell) = \partial r/\partial \ell$ , but maintained the first which is what Keynes suggested should be done. However this interpretation is inappropriate in this context. The employee takes, $\bar{u}$ as given ex ante and $r(\ell)$ as given ex post but not the wage rate.

The first best contract also dissassociates the marginal rate of substitution from the marginal rate of remuneration. In particular in the first best contract $f_1(\ell,.) = s(r,\ell) \geq \partial r/\partial \ell$ but this in no way reflects suboptimal levels of employment. Notice that in the first best contract the marginal product of labour exceeds the marginal cost of hiring labour. Therefore if it is attempted to implement the first best contract when the employee cannot observe the true value of $\theta$ the employer will wish to hire labour up to the maximum level and hence overemployment will result. The optimal incentive compatible contract may be seen as amelioratin[9] but not illiminating this effect.

Consider again equation (3.35). It is difficult to know how the divergence between the marginal product and the marginal rate of substitution varies as $\theta$ varies because of the complicated relationship between $p(\theta)$ and $g(\theta)$. However at the endpoints of the distribution there is no productive inefficiency. This is obvious if $g(a) = g(b) > 0$. However, if $g(a) = g(b) = 0$ l'hopitals rule can be applied[8] to show that $f_1(a) - s(a) = f_1(b) - s(b) = 0$ providing that $f_{12}(\ell(\theta),\theta)$ is bounded.

Corollary 2 : If and only if labour is neither a normal nor inferior good, i.e. $u_{11}s + u_{12} = 0$ then the optimal incentive compatible contract $\delta^*$ for P.3.4 is a first best contract.

Proof: Necessity: Given $(u_{11}s + u_{12}) = 0$, proceed along the lines of Lemma 3 to show that both $p(\theta) > 0$ or $p(\theta) < 0$ gives a contradiction.

Sufficiency: Given $(u_{11}s + u_{12}) \neq 0$, proceed along the lines of Lemma 3 to show that $p(\theta) = 0$ gives a contradiction.

Therefore $p(\theta) = 0$ for all $\theta \in (a,b)$ so that the incentive compatibility constraint does not bind. Hence $\delta^*$ is a first best contract .

Notice that if labour is neither a normal nor inferior good the employees utility function can be written as

$$u(r,\ell) = \hat{u}(r - h(\ell)) \tag{3.49}$$

where $\hat{u}$ is concave and h is convex. Corollary 2 provides the justification for the assumption made in chapter 1 that the employee utility function belonged to this class of functions. Hence the incentive compatibility problem could be conveniently ignored in that chapter.

Corollary 3: Given the assumptions A.3.1 and A.3.6 then the optimal incentive compatible contract $\delta^{**}$ that solves P.3.4 is unique. In addition if $\ell(\theta) = 0$ over the interval $(\theta',\theta'')$ then

$$\int_{\theta'}^{\theta''} (f_1(\theta) - s(\theta))g(\theta)d\theta + p(\theta'')(f_1(\theta'')-s(\theta''))-p(\theta')(f_1(\theta')-s(\theta')) = 0 \tag{3.50}$$

Proof 1 : To show that $\delta^{**}$ is unique it is necessary to show that $p(\theta) \geq 0$ for all $\theta \in (a,b)$. Let $q(\theta)$ be the multiplier for equation (3.35) in P.3.4 then the first order conditions for P.3.4 are

$$-\dot{p}(\theta) = (1 - \lambda u_1(f(\ell(\theta),\theta) - \pi(\theta),\ \ell(\theta)))g(\theta) \tag{3.37'}$$

$$\begin{aligned} -\dot{q}(\theta) = {} & \lambda(u_1(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))f_1(\ell(\theta),\theta) \\ & + u_2(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta)))g(\theta) + p(\theta)f_{12}(\ell(\theta),\theta) \end{aligned} \tag{3.38'}$$

$$p(a) = p(b) = q(a) = q(b) = 0. \tag{3.39'}$$

By the complementary slackness condition, for those intervals of (a,b) for which $\ell(\theta) > 0$, $q(\theta) = 0$ so that the analysis of Lemma 3 applies unammended to these regions. Suppose then there is some typical region $(\theta',\theta'')$ for which $\ell(\theta) = 0$. It has already been shown that $p(\theta') \geq 0$ and $p(\theta'') \geq 0$. In addition

$$(1 - \lambda u_1(\theta))/\partial\theta = 0 \qquad \forall\, \theta \quad \epsilon(\theta',\theta'') \tag{3.51}$$

Therefore let $(1-\lambda u_1(\theta)) = c$ for all $\theta \varepsilon(\theta', \theta''')$. Suppose that $p(\theta''') < 0$ for some $\theta''' \varepsilon(\theta',\theta'')$ then integrating equation (3.37)

$$p(\theta''') = p(\theta') - c( (\theta''') - G(\theta''))< 0 \tag{3.52}$$

and

$$p(\theta''') = p(\theta'') + c(G(\theta'') - G(\theta''')) < 0 \tag{5.53}$$

where G is the distribution function. Since G is monotone increasing equation (3.52) and (3.53) imply $c > 0$ and $c < 0$ respectively, which is a contradiction. So $p(\theta) \geq 0$ for all $\theta \varepsilon(a,b)$ and $\delta^{**}$ is unique.

2. To show that equation (3.50) holds integrate equation (3.38) for $\theta \varepsilon(\theta',\theta'')$

$$- q(\theta) = \int_{\theta'}^{\theta} (\lambda(u_1(\hat{\theta})f_1(\hat{\theta}) + u_2(\hat{\theta}))g(\hat{\theta}) + p(\hat{\theta})f_{12}(\hat{\theta}))d\hat{\theta} \tag{3.54}$$

$$= \int_{\theta'}^{\theta}((f_1(\hat{\theta}) - s(\hat{\theta}))g(\hat{\theta}) + \dot{p}(\hat{\theta})(f_1(\hat{\theta}) - s(\hat{\theta})) + p_1(\hat{\theta})f_{12}(\hat{\theta}))d\hat{\theta} \tag{3.55}$$

where equation (3.55) is obtained from equation (3.54) by using equation (3.37'). The term $p(\theta)(f_1(\theta) - s(\theta))$ can be integrated by parts remembering that $\dot{\ell}(\theta) = 0$. Therefore

$$- q(\theta) = \int_{\theta'}^{\theta}(f_1(\bar{\theta}) - s(\bar{\theta}))g(\bar{\theta})d\hat{\theta} + p(\bar{\theta})(f_1(\bar{\theta}) - s(\hat{\theta}))\Big|_{\theta'}^{\theta} . \tag{3.56}$$

Since $q(\theta'') = 0$ by the complementary slackness condition, equation (3.50) follows directly from equation (3.56) .

Corollary 3 shows that the only difference between P.3.4 and P.3.4$'$ occurs when $\dot{\ell}(\theta) = 0$. It is shown that the contract $\delta^{**}$ is unique but it cannot be concluded that there is overemployment in every state. There may be underemployment for some states for which $\dot{\ell}(\theta) = 0$. Indeed if $\dot{\ell}(\theta) = 0$ for all $\theta \varepsilon(a,b)$ equation (3.50) shows that there is some unique $\theta^*$ such that

$$f_1(\theta) - s(\theta) \leq 0 \qquad \theta \ \varepsilon(a,\theta^*) \tag{3.57}$$

$$f_1(\theta) - s(\theta) \geq 0 \qquad \theta \ \varepsilon(\theta^*,b) \tag{3.58}$$

Thus for the constant contract $\bar{\delta} = (\int_a^{\theta} f_2(\bar{l},\theta), \bar{l})$, there will be underemployment at high values of $\theta$ and over employment at low values. Notice however that the constant contract $\bar{\delta}$ allocates risk efficiently because the marginal utility of income is maintained at a constant level, as is total utility.

Until.now it has been assumed that the employer is risk neutral. Suppose hwoever that A.3.2 is modified so that

A.3.2′ $\quad v = v(\pi) : R \to R$ , is $C^2$ and $v'(\pi) > 0$ $v''(\pi) < 0$

Assumption A.3.2′ states that the employer is strictly risk averse. Then the optimal contract $\delta^{***}$ solves the following programming problem.

P.3.4″
$$\max_{\delta} \int_a^b v(\pi(\theta))g(\theta)d\theta$$

$$\text{s.t} \quad \int_a^b u(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))g(\theta)d\theta = \bar{u} \qquad (3.34)$$

$$\dot{\pi}(\theta) = f_2(\ell(\theta),\theta). \qquad (3.27)$$

Notice that the incentive compatibility constraint is unchanged because any monotonic transformation leaves the maximum the same in P.3.3. Then the following interesting result has been proved by Grossman and Hart.

Corollary 4. Given assumptions, A.3.1, A.3.2′, A.3.3-A.3.6 and if labour is neither a normal nor inferior good then $\delta^{***}$ is unique and $f_1(\theta) > s(\theta)$ for all $\theta \in (a,b)$.

Proof: The first order conditions for P.3.4 are

$$-\dot{p}(\theta) = (v'(\pi(\theta)) - \lambda u_1(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta)))g(\theta) \qquad (3.37'')$$

$$\lambda(u_1(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))f_1(\ell(\theta),\theta) + u_2(f(\ell(\theta),\theta) - \pi(\theta),\ell(\theta))g(\theta)$$

$$= -p(\theta)f_{12}(\ell(\theta),\theta) \qquad (3.38'')$$

$$p(a) = p(b) = 0 \qquad . \qquad (3.39'')$$

Then let $p(\theta) \geq 0$ over some interval $(\theta', \theta'')$. This implies $f_1(\theta) \leq s(\theta)$ via equation (38″). Therefore $u_{11}(\theta)f_1(\theta) + u_{12}(\theta) \geq u_{11}(\theta)s(\theta) + u_{12}(\theta) = 0$ since labour is neither normal nor inferior. This then implies

$$\partial(v'(\theta) - \lambda u_1(\theta))/\partial\theta = v''(\theta)f_2(\theta) - \lambda(u_{11}(\theta)f_1(\theta) + u_{12}(\theta))\,\ell(\theta) < 0 \quad \forall\ \theta\ \varepsilon(\theta', \theta'') \tag{3.59}$$

Then upon integration equation (3.37″) provides a contradiction. Therefore $p(\theta) < 0$ for all $\theta\ \varepsilon(a,b)$ and hence $f_1(\theta) > s(\theta)$ for all $\theta\ \varepsilon(a,b)$. The uniqueness of $\delta^{***}$ follows because $p(\theta)$ is uniformly signed which is all that is required in Takayma's theorem .

Corollary 4 shows that if labour is neither a normal nor inferior good and the employer is risk averse then the marginal product of labour will exceed the marginal rate of substitution between income and labour in each state. In other words there is underemployment. It is interesting to compare and contrast Corollary 1 with Corollary 4. Equation (3.59) explains the difference between the two results. In Corollary 4 equation (3.59) is assumed to be positive. In fact, in general the sign of equation (3.59) is very complicated and it may also change sign. Equally equation (3.59) does not directly determine the sign of the costate variable $p(\theta)$ which also depends on the normality of labour. It is clear for example that Corollary 4 still obtains if labour is an inferior good. Hence it is best to think of Corollary 1 and Corollary 4 as delineating circumstances when an unambiguous result can be obtained. One's choice can be made upon taste and situation.

The risk neutrality of the employer is usually justified by an appeal to a Knightian distinction between entrepreneurs and workers or by appeal to casual empirism that suggest emplyers are more wealthy or have better access to financial markets than workers or that firms are widely held by diverse individuals. However firms do take part in risk reducing activities and their purchases of insurance do not seem less than for individuals or workers. Equally firms cannot diversify against collective risks such as a 'world recession' or perhaps even against sectoral risks. This seems to suggest that employers should be modelled as risk averse. On the other hand the contract market is still a competitive market, and will tend to drive out risk averse employers. By the same token the normality of leisure as a good is widely acceptable. If for example there is no income effect in the supply of labour, the labour supply curve will always be upward sloping. Conclusions based on such an assumption can clearly be erroneous. Therefore it is perhaps true that Corollary 1 has wide applicability.

A simple example of an incentive compatible contract is given below. It illustrates the results of Lemma 3. In particular it is shown how the optimal incentive compatible contract responds to changes in the distribution of the set of states.

Example : The production, utility and density function satisfy

$$u(r,\ell) = 2\sqrt{r} - \ell : f(\ell,\theta) = \theta\ell : g(\theta) = \left\{ \begin{matrix} a & \text{if} & \theta = \theta_1 \\ 1-a & & \theta = \theta_2 \end{matrix} \right\} \quad (3.60)$$

Thus the density function has a Bernoulli distribution where

$$E\theta = a\theta_1 + (1-a)\theta_2 = \mu. \quad (3.61)$$

$$\mathrm{Var}\theta = a(1-a)(\theta_1-\theta_2)^2 = (a/(1-a)(\mu-\theta_1)^2 \quad (3.62)$$

Then if $\mu$ and $\theta_1$ are taken as given an increase in a represents a mean preserving increase in variance, that is

$$\partial(\mathrm{Var}\theta)\partial a = (\mu-\theta_1)|/(1-a)^2 \geq 0 . \quad (3.63)$$

This will be used to examine how the contract changes as the variance of the distribution changes but with the mean held constant. Such a procedure is only meant to be illustrative since the Bernoulli distribution is not really suited to a mean-variance treatment.

There are two incentive compatibility constraints for this distribution namely, that the employer cannot increase his profits by misreporting $\theta = \theta_1$ when in fact $\theta = \theta_2$ and vice-versa

$$\theta_1 \ell_1 - r_2 \geq \theta_1 \ell_2 - r_2 \tag{3.64}$$

$$\theta_2 \ell_2 - r_2 \geq \theta_2 \ell_1 - r_1 \tag{3.65}$$

where $r_j$ and $\ell_j$ are the remuneration and labour input levels in state $j = 1, 2$. Notice that if equation (3.64) holds as an equality then equation (3.65) holds as an inequality and vice-versa. That is only one of the incentive compatibility constraints binds at any one time. It will be shown that only equation (3.64) binds. Intuitively this is plausible because the first best contract will automatically satisfy equation (3.63) but not equation (3.64).

The optimal incentive compatible contract $\delta^*$ solves the following problem.

P.3.5
$$\max_{\delta = \{r_1, r_2, \ell_1, \ell_2\}} \quad a(\theta_1 \ell_1 - r_1) + (1-a)(\theta_2 \quad - r_2)$$

$$\text{s.t} \quad a(2\sqrt{r_1} - \ell_1) + (1-a)(2\sqrt{r_2} - \ell_2) = \bar{u}$$

$$\theta_1(\ell_1 - \ell_2) - (r_1 - r_2) \geq 0$$

$$\theta_2(\ell_2 - \ell_1) - (r_2 - r_1) \geq 0 \; .$$

Let the Lagrangian multipliers for the three constraints be $\lambda$, $p_1$ and $p_2$ respectively. Then the first order conditions for P.3.5 are

$$a\theta_1 - \lambda a + p_1\theta_1 - p_2\theta_2 = 0 \; . \tag{3.66}$$

$$(1-a)\theta_2 - \lambda(1-a) - p_1\theta_1 + p_2\theta_2 = 0 \; . \tag{3.67}$$

$$- a + \lambda a r_1^{-\frac{1}{2}} - p_1 + p_2 = 0. \tag{3.68}$$

$$- (1-a) + \lambda(1-a)r^{-\frac{1}{2}} + p_1 - p_2 = 0. \tag{3.69}$$

$$p_1(\theta_1(\ell_1 - \ell_2) - (r_1 - r_2)) \geq 0 \text{ with c.s.} \tag{3.70}$$

$$p_2(\theta_2(\ell_2 - \ell_1) - (r_2 - r_1)) \geq 0 \text{ with c.s.} \tag{3.71}$$

Notice from equations (3.66) and (3.67) that $\lambda = \mu$ which is set equal to unity for convenience. Since equation (3.70) and (3.71) imply $p_1 , p_2 \geq 0$ with complementary slackness it is evident from equation (3.66) or (3.67) that $p_2 = 0$ and

$$p_1 = a(1-\theta_1)/\theta_1 = (1-a)(\text{var}\theta)/(1-\theta_1)\theta_1$$
$$= (\text{var}\theta)/\theta_1(\theta_2-\theta_1) \qquad (3.72)$$

Therefore using equations (3.66) and (3.68) and (3.67) and (3.69)

$$r_1 = \theta_1^2$$
$$(\theta_2 r_2^{-\frac{1}{2}}-1) = - p(\theta_2-\theta_1)/(1-a)$$
$$= - (\text{var}\theta)/(1-a)\theta_1 \qquad (3.74)$$

The L.H.S. of equation (3.74) represents the divergence between the marginal product of labour $\theta_2$, and the marginal rate of substitution between income and labour $r^{\frac{1}{2}}$, in state two. It shows that the marginal product is less than the marginal rate of susbstitution or that there is overemployment in state two. Equation (3.73) on the other hand shows that the contract is productively efficient in state one. Differentiating the L.H.S. of equation (3.74) for a fixed value of $\mu$ and $\theta_1$ with respect to a gives

$$\partial(\theta_2 r_2^{-\frac{1}{2}} - 1)/\partial a\big|_{\mu,\theta} = -[(1-a)(\partial\text{var}(\theta)/\partial a) + \text{var}\theta]/(1-a)^2\theta_1 \leq 0. \qquad (3.75)$$

This shows that the deviation between the marginal product and the marginal rate of substitution decreases as the probability that state one occurs increases. At first sight this result seems rather counter-intuitive. It might be expected that as the probability that state two occurs decreases the employees income in that state is raised to maintain a constant level of expected utility, hence raising the marginal rate of substitution still further above the marginal product.

In fact it is known that in the general case, the transversality conditions, equation (3.39) ensure that the optimal incentive compatible contract is productively efficient at the end points of the distribution. The reason for this appears to be that by definition, the contract at the end points is constrained by incentive comptaibility on only one side. But the only reason the contract is productively inefficient is to meet the incentive comptaibility constraints. Therefore any inefficiency at the endpoints is simply a dead-weight loss. The same principles may apply to equation (3.75). For example as the probability that state two occurs falls the incentive compatibility constraint becomes less important so that productive inefficiency can be reduced. This is all very tentative and the really important point to notice is that the optimal incentive compatible contract actually depends on the probability density function $g(\theta)$. This is important because the first best contract does not depend on g though it may depend upon the parameters of the distribution through the expected utility constraint. The dependence of the optimal incentive compatible contract upon the density function will be important in any macroeconomic or general equilibrium context.

The next section presents some concluding remarks.

## Section 3 : Conclusions

This chapter has shown how the optimal labour contract is affected if the contracting parties have asymmetric information. It has been shown that the optimal incentive compatible contract will be both productively inefficient and also share risk inefficiently. In general if the employee has better information there will be underemployment and if the employee has better information there will be overemployment.

It was shown that the optimal incentive compatible contract depends ipon the distribution function of the state of nature. This is important because the first best contract does not depend upon the density function. This feature of the optimal incentive compatible contract will be used in chapter 4 to examine optimal contracts when there is imperfect information.

The two cases of asymmetric information studied in this chapter are not the only instance of information asymmetry in labour contracts. For example when considering a potential employee, the employer might not know the distribution function $g(\theta)$ that indicates whether the employee is on average a good or bad worker. To be more specific the density function may be parameterized $g(\theta,\alpha)$ where $\alpha$ is known to the potential employee but not to the employer. A situation of this type has been studied by Rothschild and Stiglitz (1976) in the context of an insurance market. Alternatively the situation might be reversed so that the employer knows $\alpha$ but the employee does not. This would appear to be a fairly typical and interesting information asymmetry which would make an interesting extension to the present model.

Notes

1. A result of this kind is given by Akerlof (1970).
2. Such a contract is pareto efficient.
3. Green and Kahn only examine the second case. The first case has also been studied by Cooper (1981).
4. See Holmstrom (1981) for a preliminary investigation.
5. It is at this point that a concern for reputation becomes important.
6. It is assumed that the non-negativity constraint on the supervisors income is not binding, or alternatively X is sufficiently large.
7. Again profits are always assumed to be non-negative. An examination of this constraint is given by H. Grossman (1977).
8. See Stiglitz (1977).

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