Dynamic Relational Contracts under Complete Information

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Abstract

This paper considers a long-term relationship between two agents who both undertake an action or investment that produces a joint benefit. Agents have an opportunity to expropriate some of the joint benefit for their own use. Agents have quasi-linear preferences. Two cases are considered: where agents are risk averse but where limited liability constraints do not bind, and where agents are risk neutral and subject to limited liability constraints. We ask how to structure the investments and division of the surplus over time to avoid expropriation. In the risk-averse case, the dynamics of actions and surplus may or may not be monotonic depending on whether or not a first-best allocation can be sustained. Agents may underinvest but never overinvest. If the first-best allocation is not sustainable, there is a trade-off between risk sharing and surplus maximization; surplus may not be at its constrained maximum even in the long run and the "amnesia" property of pure risk-sharing models fails to hold. In contrast, in the risk-neutral case there may be an initial phase in which one agent overinvests and the other underinvests. Both actions and surplus converge monotonically to a stationary state, where surplus is maximized subject to the self-enforcing constraints.

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1. Introduction

This paper considers a situation where two agents repeatedly engage in joint production. In each period, both agents simultaneously undertake an action or investment that produces a joint output. Agents must also decide how to share the joint output each period. We assume there is a hold-up problem, that is, contracts on actions or the division of the joint output are not enforceable and in addition the outside option of each agent is increasing in the investment of the other agent. We allow joint output and the outside options of the agents to depend on an exogenous state. We consider cases where the agents are risk averse and where they are risk neutral. The only link between periods is a Markov process

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determining states. There is complete information: apart from the fact that the agents choose their actions simultaneously each period, everything is observable. The only friction is that contracts cannot be enforced. We consider allocations or contracts from which no agent has an incentive to renege by imposing self-enforcing constraints at each date and state. We refer to feasible contracts that satisfy these constraints as dynamic relational contracts. We characterize the Pareto-efficient dynamic relational contracts; we refer to such contracts as optimal contracts.

We impose two simplifying assumptions on our model. First, we assume that agents' preferences are quasi-linear in consumption and actions. This simplifies the problem because with quasi-linear preferences efficient actions (and hence, surplus) are determined independently of the distribution of resources (the marginal rate of substitution between consumption and the action is equal to unity). Second, we impose sufficient conditions such that the constrained Pareto-frontier is concave. This simplifies our problem because it allows us to focus on non-random contracts.[1] We examine two main cases: where agents are risk averse but preferences are such that non-negativity constraints on consumption can be ignored, and where agents are risk neutral but consumption is constrained to be non-negative (limited liability).

If agents are risk averse results depend on whether or not it is possible to sustain a first-best allocation for some division of the surplus. If it is possible, convergence to the first best is monotone. Otherwise there might be an initial monotone phase, but in the long-run, when there are two or more states, monotonicity does not generally obtain: when the same state recurs, surplus will sometimes be higher at the later date and sometimes lower. There is also a trade-off between achieving efficient risk-sharing and maximizing current surplus even in the long run. In particular, and in contrast to the risk-neutral case, current surplus is not maximized. Better risk-sharing is achieved by holding the action of one agent inefficiently low because this reduces the outside option of the other agent, that is, it relaxes the latter's self-enforcing constraint. We show that the optimal contract depends on the past history of states and so the "amnesia" property of the risk-sharing limited commitment model does not hold.

When agents are risk neutral, we consider the implications of limited liability constraints and show that optimal contracts involve two phases. In the first phase there is backloading with zero consumption for the constrained agent, who overinvests up to the last period of the backloading phase and the terms of the contract move monotonically in his/her favor. This overinvesment arises because it allows a further transfer of utility to the other agent who consumes the extra output. It occurs despite the hold-up problem, that in a static model would lead to underinvestment. Nevertheless, we demonstrate that because of backloading it is never the case that both agents overinvest—even at different dates—in any optimal contract. The second phase is stationary and independent of the initial conditions. Consumption and investment depend on the state but not on the time period.

[1] It would be straightforward to allow for random contracts by introducing a public randomization device, but at the cost of considerable complexity of notation and statements of our results. Furthermore, the assumptions we make are consistent with those that are commonly made in the literature.

Each agent has positive consumption and, for a given state, either both invest efficiently or both underinvest. In either case, current surplus is maximized subject to the self-enforcing constraints. Convergence to the stationary phase is monotone in the sense that whenever the same state recurs in the backloading phase, surplus is higher at the later date.

Related literature

A number of results for special or limiting cases of this model are known. First, one-sided-action versions of this model or variations on it, have been studied by a number of authors (see, e.g., Thomas and Worrall 1994, Sigouin 2003, Albuquerque and Hopenhayn 2004, Kovrijnykh 2013). Typically, this literature has considered the case where both agents are risk neutral, there is limited liability and the agent taking the action can commit. To prevent the uncommitted agent from taking his/her outside option, actions may be kept low initially. A key insight of this literature is that incentives are improved when payments to the uncommitted agent are backloaded into the future. This provides a growing carrot for adhering to the contract. Consequently, the action or investment of the other agent can be increased in the future. This generates dynamics in the agent's actions as well as in monetary payments. In the long run, actions and transfers converge to a stationary distribution that maximizes the surplus, output less action costs, given the self-enforcing constraints. The speed of backloading is restricted by the limited liability constraints. Ray (2002) has established the most general backloading result of this type. He considers a general, but non-stochastic, principal-agent model in which both parties may take actions. The principal can commit within each period, so the self-enforcing constraint only applies to the agent. He shows that an efficient contract has terms that move in favor of the agent, converging in finite time to the efficient self-enforcing continuation that maximizes the agent's payoff. Our results generalize this backloading result to the case where both agents undertake an action and neither agent can commit. Furthermore, we demonstrate that there may be overinvestment in the risk-neutral case, and in the risk-averse case show that there may be a trade-off between productive efficiency and risk sharing even in the long run. Neither of these properties occur in models where only one agent takes an action.

Second, consider the case where agents have no action to take, or where there is no hold-up problem. In this case, the model involves sharing a stochastic endowment. The case in which agents have their own stochastic endowment and can share risk subject to limited commitment constraints has been widely studied (see, e.g., Kocherlakota 1996, Ligon et al. 2002, Thomas and Worrall 1988). A result of this pure risk-sharing case is that a constrained Pareto-efficient allocation evolves toward a stationary distribution, and that, for some parameter values, the distribution of future expected utilities is non-degenerate. Although the distribution is non-degenerate, the solution exhibits an "amnesia" property that once an agent is constrained, the contract from then on is independent of the past history of shocks. With hold-up, the optimal contract depends on the past history of states and does not in general exhibit the amnesia property of the pure risk-sharing model.[2] Furthermore,

[2] Ábrahám and Lacsó (2017) establish a similar result in a model of risk-sharing model and storage. The absence of the amnesia property is more consistent with the empirical evidence (see Broer 2013).

the pure risk-sharing literature only considers distributional issues and has no implications for the efficiency and dynamics of actions that are the focus of this paper. Nevertheless, we are able to demonstrate a limit result that as our hold-up problem vanishes, the optimal contract converges to the standard pure risk sharing contract.

Third, there are a very few papers in this limited commitment literature that examine the situation where two or more agents take actions. The most relevant paper to ours is Acemoglu et al. (2011) that considers a model of changes in political power. In Acemoglu et al. (2011) a Markov process determines which risk-averse political party is in power. Political parties take actions that contribute to a common pool of resources whether in power or not, but only the party in power gets to determine the allocation of resources across agents. Therefore, states are identified by the agent in power. It is shown that in a constrained Pareto-efficient allocation, the action of one of the agents (the one in power) is always chosen efficiently and actions of other agents (those not in power) are distorted downward. Furthermore, they establish a convergence result that depends on whether a first-best allocation is sustainable or not: if a first-best allocation is sustainable, then the actions and the division of resources converges to a degenerate (first-best) distribution; otherwise, allocations converge to a non-degenerate distribution (it need not be unique). The two-agent model with quasi-linear utility considered in their paper corresponds to the limiting case of our model where in each state one agent has all the property rights. We also establish convergence results but our results apply for a general distribution of property rights and an arbitrary number of states and may result in the actions of both agents being inefficiently low, even in the long run. Their convergence result, when a first-best allocation is sustainable, corresponds to our Theorem 1(a). In Theorem 1(b), when a first-best allocation is not sustainable, we establish convergence to a unique limiting distribution that is independent of initial conditions.

Fourth, our model is related to the broader literature on relational contracting (see, e.g., Levin 2003, Doornik 2006, Rayo 2007) that builds on the work of Macleod and Malcomson (1989). This literature has studied models with more general ingredients (including many-sided actions, enforceable payments, moral hazard, hidden information, and endogenous property rights), but has restricted attention to stationary equilibria, thus, eliminating any interesting dynamics in investments and transfers. The restriction to stationary equilibria is either derived, because stationary contracts are optimal (when agents are risk neutral and in the absence of limited liability), or imposed, because the focus is on organizational structures under which full efficiency can be achieved. Most of this literature is therefore silent on the dynamics of relational contracts that are the main concern of this paper.[3]

[3]One exception to the focus on stationary contracts is Fong and Li (2017) who introduce limited liability and moral hazard into a risk-neutral model firms and workers based on Levin (2003). They show that if the principal extracts most of the surplus, the backloading of the agent's utility can lead to a probationary contract in which the agent's wage is initially at the lower bound, and incentives are provided by the threat of termination; at some point this threat is removed and the wage increases to a higher level.

Illustrative example

To illustrate the model we have in mind, we present a simple example with no uncertainty and risk averse agents.[4] There are two agents with common discount factor $\delta \in (0, 1)$ and an infinite horizon. In each discrete period the action or effort of agent i is a_i and joint output is additive

$$y(a_1, a_2) = f_1(a_1) + f_2(a_2) = 2\left(\sqrt{a_1} + \sqrt{a_2}\right).$$

Both agents have common preferences satisfying constant absolute risk aversion with coefficient 1/2:

$$u_i(x) = 2\left(1 - \mathrm{e}^{-\frac{1}{2}x}\right),$$

where $x := c - a$, consumption less effort. Actions take place simultaneously at the beginning of each period. At the end of the period, output is realized and it is divided between the two agents. Suppose, that irrespective of how output is divided, agent i can unilaterally get a breakdown consumption of $\phi_i(a_1, a_2) = \theta_{i1} f_1(a_1) + \theta_{i2} f_2(a_2)$, that depends on the action of the other agent. For parameters $\theta_{11} = \theta_{22} = 0$ and $\theta_{12} = \theta_{21} = 1$, this means that either agent can expropriate all of the other agent's output but if they do so they lose their own output. A relational contract is just an agreed sequence of actions and division of the output from these actions from which no agent has an incentive to deviate. We assume that if a deviation occurs, in each period thereafter the agents revert to short-run Nash equilibrium anticipating the breakdown payoffs $\phi_i(a_1, a_2)$. With the specification for θ_{ij} just given, the short-run Nash equilibrium has $a_i, c_i = 0$ (and hence, $u_i = 0$) and the discounted payoff from a deviation, the deviation utility, is:

$$D_i(a_j) = u_i\left(2\sqrt{a_j}\right) = 2\left(1 - \mathrm{e}^{-\sqrt{a_j}}\right).$$

We characterize *constrained* Pareto optimal contracts, that is, within the set from which no agent would deviate. At the first best, $a_1^* = a_2^* = 1$ and surplus $z := y(a_1^*, a_2^*) - a_1^* - a_2^* = 2$ is maximal. This is sustainable provided an equal split of surplus ($x_i^* = 1$) is an equilibrium:

$$\frac{u_i(1)}{(1-\delta)} = \frac{2(1 - \mathrm{e}^{-\frac{1}{2}})}{(1-\delta)} \geq D_i(1) = 2(1 - \mathrm{e}^{-1}),$$

or $\delta \geq (1 + \sqrt{\mathrm{e}})^{-1}$.

Suppose that $\delta \geq (1 + \sqrt{\mathrm{e}})^{-1}$, so that the first-best allocation is sustainable. Let V_i denote the lifetime utility of agent i. For $\delta > (1 + \sqrt{\mathrm{e}})^{-1}$ surplus will be constant at its efficient level for a range of values for V_1. Consider starting from a feasible value of V_1 below $D_1(1) = u_1(2)$, i.e., worse for agent 1 than the deviation utility at the first-best allocation.

[4]For the purpose of constructing a simple example that illustrates the solution, we here ignore the non-negativity constraints on consumption. We use parameter values such that the Pareto-frontier is concave. In the Supplementary Material, we show how to fully solve this example using our characterization results and without having to use value function iteration.

If $a_2 = a_2^*$, then agent 1 would deviate. Therefore, $a_2 < a_2^*$; the best contract has a_2 as high as possible, such that agent 1 does not wish to deviate, i.e., $V_1 = D_1(a_2)$. Since we are assuming $\delta \geq (1+\sqrt{\mathrm{e}})^{-1}$, for the corresponding value of V_2 on the Pareto frontier, $V_2 > u_2(2)$. That is, agent 2 is unconstrained and $a_1 = 1$ (is efficient). Since a_2, and hence, surplus z, is determined by the binding constraint $V_1 = D_1(a_2)$, both can be expressed as functions of V_1. Hence, in this example:

$$\begin{aligned}
a_1(V_1) &= 1, \\
a_2(V_1) &= \left(\tfrac{1}{2}u_1^{-1}(V_1)\right)^2 = \left(\log\left(1-\tfrac{V_1}{2}\right)\right)^2, \\
z(V_1) &= 1 + u_1^{-1}(V_1) - \left(\tfrac{1}{2}u_1^{-1}(V_1)\right)^2 = 1 - 2\log\left(1-\tfrac{V_1}{2}\right) - \left(\log\left(1-\tfrac{V_1}{2}\right)\right)^2, \\
z'(V_1) &= u_1^{-1\prime}(V_1)\left(1 + u_1^{-1}(V_1)\right) = \left(1+\log\left(1-\tfrac{V_1}{2}\right)\right)\left(1-\tfrac{V_1}{2}\right)^{-1}.
\end{aligned}$$

It is easily checked that $z(V_1)$ is increasing and concave in this region with $z(0) = z'(0) = 1$ and $z(u_1(2)) = 2$ and $z'(u_1(2)) = 0$. We show below (equation (4.1) in Section 4) that for values of V_1 where the surplus is increasing in V_1, as here for $V_1 < u_1(2)$, then V_1 will be higher next period. It follows straightforwardly that V_1 is an increasing sequence converging to $u_1(2)$. So, the contract converges to the surplus maximizing actions, here the first best. We also show below (see equation (3.2b) in Section 3) that surplus is divided so that u_2'/u_1' is equal to the absolute value of the slope of the (strictly concave) Pareto frontier in the following period. Since V_1 is increasing over time, so too is u_2'/u_1'.[5] That is, the way the surplus is distributed (as well as V_1) moves monotonically in favor of agent 1. Thus, backloading of agent 1's utility occurs, and in such a way as to guarantee efficiency in the long-run.

The case where $\delta < (1+\sqrt{\mathrm{e}})^{-1}$ is similar. There is convergence to a stationary value of V_1. However, in this case, convergence of V_1 is to a point where the actions maximize the joint surplus subject to the no-deviation constraints. At this value of V_1, and for some neighborhood around it, both constraints $V_i = D_i(a_j)$ bind.

Taking both cases together, it can be concluded that with no uncertainty there is convergence to the constrained surplus maximizing actions for any δ.

Plan of paper

The paper extends this example to consider a more general production function and breakdown payoffs. We consider multiple states and quasi-linear preferences including risk-neutrality and non-negativity constraints on consumption. We show that the convergence result of the example generalizes to the case with multiple states when the first-best is sustainable (and also when agents are risk-neutral), but otherwise, with risk aversion and multiple states there is a trade-off between risk sharing and efficiency and convergence to surplus maximization does not occur. In this case we show that u_2'/u_1' converges to a non-degenerate limiting distribution independent of the initial distribution of the surplus.

[5] Convergence of u_2'/u_1' is to $\mathrm{e}(1-\delta(1-\mathrm{e}))^{-2} \leq 1$; convergence is to 1 for $\delta = (1+\sqrt{\mathrm{e}})^{-1}$.

The paper proceeds as follows. Section 2 describes the model. Section 3 provides some general results that apply to both the risk-neutral and risk-averse cases. Section 4 analyzes the risk-averse case and Section 5 the risk-neutral case. Section 6 concludes. Statements of lemmas and the proofs of theorems are found in the Appendix. Proofs of Propositions and Lemmas are relegated to the Supplementary Material.

2. Model

We consider a dynamic model of joint production where agents repeatedly undertake an action or investment that generates a joint output. There is no asset accumulation and full depreciation of the investment in each period. Once produced agents have the opportunity to unilaterally expropriate some of the joint output for their own benefit. In this section, we shall describe the economic environment and the set of dynamic relational contracts. We define a game played by the two agents and identify dynamic relational contracts with the subgame perfect equilibria of that game. Our interest is in optimal contracts, that correspond to the set of Pareto-efficient subgame perfect equilibria.[6]

2.1. Economic environment

Time is discrete and indexed by $t = 0, 1, 2, \dots, \infty$. At the start of each period, a state of nature s is realized from a finite state space $\mathcal{S}$ with $n \geq 1$ states. The state evolves according to an irreducible, time homogeneous Markov chain with transition matrix $[\pi_{sr}]$, where $\sum_{r \in \mathcal{S}} \pi_{sr} = 1$, all $s \in \mathcal{S}$. The chain starts from an initial state s_0 at date $t = 0$. We denote the state at date t by s_t and the history of states by $s^t = \{s_0, s_1, \dots, s_t\}$.

There are two agents, $i = 1, 2$. At every date t, and after the state at that date is observed, both agents simultaneously choose an action/investment $a_i \in \mathbb{R}_+$. Actions produce an output $y^s(a) \geq 0$ that depends on the state s and the action pair $a := (a_1, a_2)$ (details are given below in Assumption 2). Having observed actions and output, the agents agree to split output and each consumes non-negative consumption c_i, $c := (c_1, c_2) \in \mathbb{R}^2_+$. We impose that consumption is non-negative as a simple way to reflect a limited liability constraint on the transfers one agent can make to the other. Consumption c is *feasible* if $c_1 + c_2 \leq y^s(a)$. Agent i derives per-period utility u_i from *net consumption* $x_i := c_i - a_i$, $x := (x_1, x_2) \in \mathbb{R}^2$. We make the following assumptions on u_i and y^s:

Assumption 1. Per-period utility $u_i \colon [\underline{x}_i, \infty) \to \{-\infty\} \cup \mathbb{R}$ is a twice continuously differentiable, strictly increasing and concave function of net consumption, where $\underline{x}_i \leq 0$.

Assumption 2. For each $s \in \mathcal{S}$, the production function $y^s \colon \mathbb{R}^2_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly increasing in both arguments and strictly concave. Furthermore, for each $s \in \mathcal{S}$, $\partial^2 y^s(a)/\partial a_1 \partial a_2 \geq 0$ (complementarity); $y^s(0) = 0$ and the upper contour sets $\{a \in \mathbb{R}^2_+ \mid y^s(a) - a_1 - a_2 \geq \gamma\}$, $\gamma \in \mathbb{R}$, are compact.

[6] More precisely, we focus on efficient pure subgame-perfect equilibria relative to specified "Nash reversion" punishments, although our characterization also applies mutatis mutandis to optimal punishments, should they be different, and hence, to efficient equilibria among the set of all pure strategy equilibria.

Assumption 2 imposes fairly standard conditions on the production function. The last part of Assumption 2 is a simple way to restrict actions to a compact set $\mathcal{A}(s)$. Denote *surplus* in state s by $z^s(a) := y^s(a) - a_1 - a_2$. Define the *first-best action pair* $a^*(s)$ as the actions that maximize surplus in state s. Given Assumption 2, the first-best action pair exists and is unique. We refer to the surplus $z^s(a^*(s))$ as the *first-best surplus*. Since actions are chosen simultaneously and independently, we also define the *conditionally efficient actions* $a_i^*(a_j, s)$, $i, j = 1, 2$, $i \neq j$, such that

$$a_i^*(a_j, s) := \arg\max_{a_i \in \mathbb{R}_+} \; [y^s(a_1, a_2) - a_i].$$

The conditionally efficient actions are single-valued, continuous functions of the other agent's action.[7] The weak complementarity assumption is slightly restrictive but reflects our view that relational contracting framework is most natural when there are complementarities in production. Given the weak complementarity assumption, conditionally efficient action functions are weakly upward sloping. In addition, $a_i^*(s) = a_i^*(a_j^*(s), s)$ for $i, j = 1, 2$, $i \neq j$.

We now specify what an agent can get if there is no agreement on how to divide up output. If no agreement is reached, agent i gets a *breakdown consumption* of $\phi_i^s(a)$, and hence, a *breakdown utility* of $u_i(\phi_i^s(a) - a_i)$. An agent can always take the option of receiving her breakdown utility. More formally, we suppose the agents play a Nash demand game to divide output.[8] In this Nash demand game, both agents simultaneously announce consumption claims $(\widetilde{c}_1, \widetilde{c}_2)$, $\widetilde{c}_i \geq 0$. If $\widetilde{c}_1 + \widetilde{c}_2 = y^s(a)$, then this determines the division of output: consumption $c_i = \widetilde{c}_i$. Otherwise, agents receive their breakdown consumption: $c_i = \phi_i^s(a)$.

The specific assumptions on $\phi_i^s(a)$ are given below, but a simple example with *proportional defaults* captures what we have in mind. Suppose that each agent can, by defaulting, capture a fraction θ_i of the available output $y^s(a)$. Here, $\phi_i^s(a) = \theta_i y^s(a)$. We assume that agents cannot obtain more than the available output, so $\theta_1 + \theta_2 \leq 1$. We do not require that the sum exhausts available output. For example, disagreement may incur a cost, such as lawyers' fees or bargaining costs, so that some of output is lost when there is default. In such cases, $\theta_1 + \theta_2 < 1$. We assume $\theta_i > 0$, so that what an agent gets in the breakdown is increasing in the action of the other agent. This assumption captures the hold-up feature of joint production we wish to model.

As another example, consider the special case with *additive* production: $y^s(a) = f_1^s(a_1) + f_2^s(a_2)$ and suppose $\phi_i^s(a) = \theta_{i1}^s f_1^s(a_1) + \theta_{i2}^s f_2^s(a_2)$, $\theta_{ij}^s \geq 0$ and $\sum_{i=1}^2 \theta_{ij}^s \leq 1$, $j = 1, 2$ (this is very similar to the formulation used by Halonen (2002)). Our hold-up assumption requires $\theta_{ij}^s > 0$, $i, j = 1, 2$, $i \neq j$. With this parameterization, assuming $\sum_{i=1}^2 \theta_{ij}^s = 1$ and taking

[7] This result is simple and straightforward to show. Formal statements of results of this type are stated as lemmas in the Appendix and proofs are given in the Supplementary Material. For example, the fact that conditionally efficient actions are single-valued and continuous is stated as Lemma 1 in the Appendix.

[8] This is approach is also used by Hall (2005), for example. What we want to capture is that there is an ex ante agreement on what actions should be taken, and how the resulting output should be split, and that failure to abide by it leads to the breakdown utilities. The Nash demand game is a simple way of implementing this idea – but we stress that our results are not sensitive to the way it is operationalized. For a fuller discussion, see Hall (2005).

the limit as $\theta^s_{ij} \to 0$, for $i, j = 1, 2$, $i \neq j$ and for all $s \in \mathcal{S}$, produces the pure risk sharing model that has been studied by Kocherlakota (1996), Ligon et al. (2002) and others. This is discussed in Section 4.

Analogous to Assumption 2, we shall assume that $\phi^s_i(a)$ satisfies:

Assumption 3. For each $s \in \mathcal{S}$ and $i = 1, 2$, the function $\phi^s_i\colon \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous, twice continuously differentiable, strictly increasing in both arguments and strictly concave. Moreover, $\partial^2\phi^s_i(a)/\partial a_1 \partial a_2 \geq 0$ (complementarity) and $\partial\phi^s_i(0, a_j)/\partial a_i > 1$ for all $a_j \in \mathbb{R}_+$, $i, j = 1, 2$, $i \neq j$. In addition, $\phi^s_i(0, 0) = 0$ for $i = 1, 2$ and

$$(2.1) \qquad \frac{\partial\phi^s_1(a)}{\partial a_i} + \frac{\partial\phi^s_2(a)}{\partial a_i} \leq \frac{\partial y^s(a)}{\partial a_i} \qquad \forall s \quad \text{and} \quad i = 1, 2.$$

In the case of proportional defaults, these conditions (apart from $\partial\phi^s_i(0, a_j)/\partial a_i > 1$) follow directly from Assumption 2. Complementarity in Assumption 3 implies that the reaction functions in the breakdown game are weakly upward sloping, and this simplifies the arguments below. Condition (2.1) requires that the marginal change in the total breakdown consumption from a change in the action of one of the agents cannot exceed the corresponding marginal product. Together with $\phi^s_i(0, 0) = 0$, it implies that the $\phi^s_i(a)$ are feasible, that is, $\phi^s_1(a) + \phi^s_2(a) \leq y^s(a)$ for each a and s. Condition (2.1) together with $\partial\phi^s_i(0, a_j)/\partial a_i > 1$, $i = 1, 2$ implies that the first-best action pair is strictly positive. The assumption that ϕ^s_i is strictly increasing in both its arguments, in particular that $\partial\phi^s_i(a)/\partial a_j > 0$ for $i \neq j$, captures the hold-up property of the model.

Denote the Nash best-response functions (functions because $\phi^s_i(a_1, a_2)$ is strictly concave in a_i) in the breakdown game by

$$a^N_i(a_j, s) := \operatorname*{arg\,max}_{a_i \in \mathbb{R}_+} \; [\phi^s_i(a_i, a_j) - a_i].$$

The Nash best response function $a^N_i(a_j, s)$ is continuous and weakly increasing in a_j. Moreover, we have $0 < a^N_i(a_j, s) < a^*_i(a_j, s)$ for each a_j and every state $s \in \mathcal{S}$. It is strictly positive because $\partial\phi^s_i(0, a_j)/\partial a_i > 1$ and is less than the conditionally efficient action because of the hold-up assumption that $\partial\phi^s_i(a)/\partial a_j > 0$. The best-response breakdown utility is

$$u^N_i(a_j, s) := u_i(\phi^s_i(a^N_i(a_j, s), a_j) - a^N_i(a_j, s)).$$

A Nash equilibrium of the breakdown game occurs where the best-response functions intersect (existence follows by standard arguments). Without further assumptions, the Nash equilibrium need not be unique (though it is unique if the defaults are proportional). However, the potential non-uniqueness is not critical because the Nash equilibria can be Pareto-ranked (because the best-response functions are non-decreasing and all Nash equilibria lie below the first-best action pair $a^*(s)$). Henceforth, we let $(a^{NE}_1(s), a^{NE}_2(s))$ denote the dominant Nash equilibrium and all our results apply relative to this dominant Nash equilibrium.

2.2. Dynamic relational contracts

We refer to a non-negative action and consumption sequence $\{a(s^t), c(s^t)\}_{t\geq 0}$ as a *plan*. Corresponding to a plan, agent i's lifetime utility is

$$V_i(s_0) := \mathbb{E}\left[\sum\nolimits_{t=0}^{\infty} \delta^t u_i(c_i(s^t) - a_i(s^t)) \mid s_0\right],$$

where δ is a common discount factor, $0 < \delta < 1$, and $\mathbb{E}$ denotes expectation. A plan is feasible if $\sum_i c_i(s^t) \leq y^{s_t}(a(s^t))$ for every history s^t and $c_i(s^t) - a_i(s^t) \geq \underline{x}_i$ for $i = 1, 2$ and every history s^t.

A *dynamic relational contract*, or simply *contract*, is a feasible plan from which neither agent has an incentive to deviate. The incentive to deviate depends on the punishment for deviation. This is given by the breakdown payoffs in the current period (subsequent to the deviation), and by play of the (dominant) equilibrium of the static breakdown game in all future periods. In particular, suppose that a is the current recommended action pair. If agent i is to deviate at t, then the best she can do is to choose $a_i^N(a_j(s^t), s_t)$, which yields a current payoff $u_i^N(a_j(s^t), s_t)$.[9] She is punished from $t+1$ by "Nash reversion" in which both agents choose their best responses in the breakdown game, that is, both will thereafter play the (dominant) Nash equilibrium of the breakdown game described above.[10]

Let $D_i^s(a_j)$ denote the *deviation utility*: the best discounted payoff that agent i can get by deviating, given agent j's putative action a_j in state s. It is defined recursively by

$$D_i^s(a_j) := u_i^N(a_j, s) + \delta \sum\nolimits_{r\in\mathcal{S}} \pi_{sr} D_i^r(a_j^{NE}(r)),$$

where $D_i^r(a_j^{NE}(r))$ is the deviation utility from the play of the Nash equilibrium in state r. Given our hold-up assumption (see Assumption 3), it follows that the deviation utility is continuous, differentiable, strictly increasing and strictly concave in the action of the other agent.

We stress that replacing the Nash reversion punishments by any state dependent continuation utilities that are no greater than the Nash reversion punishments leaves all the characterization results we derive intact. In particular, optimal punishments satisfy this property. Equally, if agents can take state-dependent outside options at the start of any period, then, provided these outside options satisfy the condition that they are no greater than the Nash reversion punishments, all our results apply. For example, if in periods after a default the breakdown consumptions/utilities were lower than they are in an on-going relationship, then our results still hold.

[9]Deviation at the output division stage cannot be preferable since breakdown is triggered in either case, and a_i may not be optimal in the breakdown.

[10]A dynamic relational contract is equivalent to a pure strategy subgame perfect equilibrium relative to future reversion to this Nash equilibrium. Here, strategies are infinite sequences of history-dependent actions and consumption *claims*. Punishment consisting of immediate triggering of the breakdown, and repeated play of the (dominant) Nash equilibrium of the breakdown game thereafter, is subgame perfect (each agent just *demands* the whole output after any deviation (i.e., off the equilibrium path), triggering the breakdown game each period).

Since an agent can always take the option of receiving her breakdown utility, the deviation utility provides a lower bound (as a function of the other agent's action) on the discounted utility an agent gets in any dynamic relational contract . Hence, $\{a(s^t), c(s^t)\}_{t=0}^{\infty}$ is a dynamic relational contract if it is feasible and if for every s^t, and $i, j = 1, 2$, $i \neq j$,

$$(2.2) \quad V_i(s^t) := u_i(c_i(s^t) - a_i(s^t)) + \mathbb{E}\left[\sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_i(c_i(s^\tau) - a_i(s^\tau)) \mid s^t\right] \geq D_i^{s_t}(a_j(s^t)).$$

The *continuation utility* $V_i(s^t)$ is the discounted utility that agent i anticipates from the contract after the history s^t. The right hand side of (2.2) is the deviation utility agent i gets from deviating from the recommended action after the history s^t. We refer to the inequalities (2.2) as the *self-enforcing* constraints. Whenever (2.2) holds with equality, we say that agent i is *constrained*. Otherwise, we say that agent i is unconstrained.

Dynamic relational contracts exist. For example, the *trivial* contract that has $a_i(s^t) = a_i^{NE}(s_t)$ and $c_i(s^t) = \phi_i^{s_t}(a^{NE}(s_t))$ for all s^t is both feasible and self-enforcing and therefore a dynamic relational contract. We show below (see Proposition 2) that there exist other non-trivial dynamic relational contracts.[11] Corresponding to any dynamic relational contract, $\{a(s^t), c(s^t)\}_{t=0}^{\infty}$, and initial state s_0, is a pair of lifetime utilities $(V_1(s_0), V_2(s_0))$. Given the set of dynamic relational contracts, let $\mathcal{V}_{s_0}$ denote the set of the corresponding lifetime utilities. Our objective is to characterize contracts corresponding to the Pareto-frontier of the set $\mathcal{V}_{s_0}$. We refer to dynamic relational contracts that correspond to this Pareto-frontier as *optimal contracts* and refer to the corresponding actions as *optimal actions*. We say that agent i *underinvests* (or that the action is *inefficiently low)* at some date t in an optimal contract if the optimal actions are such that $a_i(s^t) < a_i^*(a_j, s)$ and say the agent *overinvests* (or the action is *inefficiently high*) if $a_i(s^t) > a_i^*(a_j, s)$. Given the stochastic history s^t, we can treat an optimal contract as a stochastic process for (a, c). We will be interested in the long-run behavior of this process and whether it converges, and if so, whether convergence is dependent on s_0 or $V_1(s_0)$.

3. Preliminary results

This section establishes some preliminary results on the Pareto-frontier of the set of dynamic relational contracts and optimal actions. Section 4 considers the case where agents are risk averse and Section 5 will consider the case where agents are risk neutral.

3.1. Relationship to the Nash actions

Proposition 1. In any optimal contract (i) actions are never below the Nash reaction functions, $a_i(s^t) \geq a_i^N(a_j(s^t), s_t)$, and $a(s^t) \geq a^{NE}(s_t) > 0$ for all s^t; (ii) an agent who is allocated all current output and who is not overinvesting (i.e., $a_i(s^t) \leq a_i^*(a_j(s^t), s_t)$), is unconstrained.

The intuition for (i) is that if the action of one of the agents, say agent 1, were below the Nash reaction function, a Pareto improvement could be found by increasing the action of

[11] Intuitively, hold-up creates an inefficiency and provided $\delta > 0$, repeated game arguments allow cooperation to improve on the breakdown Nash equilibrium.

agent 1 by a small amount. Although the deviation utility of agent 2 increases (by hold-up), his consumption can be increased to prevent a violation of his self-enforcing constraint, and there is sufficient extra output remaining to more than compensate agent 1 for the increase in her action. This property then implies that actions can never be below the Nash equilibrium actions, $a(s^t) \geq a^{NE}(s_t)$. Since it can be shown that the Nash equilibrium actions are strictly positive, $a_i^N(a_j, s) > 0$, it follows that optimal actions are always positive too. Although (ii) is not trivial, it is unsurprising. Suppose, say, that agent 1 is allocated all of the current output. Then, agent 1 is receiving more of output than she would obtain in the breakdown game, if she held her action constant (because, by Assumption 3, agent 2 can claim a positive share of output in the breakdown game). In a deviation, agent 1 will optimize her action, but since she is not overinvesting, reducing her action to the Nash reaction function will only reduce output net of her effort. Hence, she would be worse off than receiving all output at the higher action. The continuation utility cannot be lower than the deviation continuation utility, so a deviation will lead to output being shared and a punishment continuation, worse than the equilibrium path and thus, agent 1 could not be constrained. In fact, we shall show later (see the discussion after the first-order conditions (3.2a)–(3.2c)) that any agent with positive consumption will not overinvest, and therefore the caveat in Proposition 1(ii) can be dispensed with.

3.2. Concavity, continuity and differentiability

We define $V_2^s(V_1)$ to be the Pareto-frontier of the set $\mathcal{V}_s$. It is not necessarily concave; in particular the concavity of the deviation utility $D_j(a_i)$ in the action of the other agent implies that the self-enforcing constraints (2.2) may not be satisfied at average actions and hence the constraint set need not be convex. Nevertheless, the Pareto-frontier can be shown to be concave under some additional restrictions. We state and discuss two *alternative* sufficient conditions for concavity in the Appendix, Assumption A.4 and Assumption A.5.

Proposition 2. For each $s \in \mathcal{S}$ (i) under either Assumption A.4 or Assumption A.5, $V_2^s(V_1)$ is a continuous and concave function of V_1 defined on a non-degenerate closed interval $[\underline{V}_1^s, \bar{V}_1^s]$, and is continuously differentiable on its interior. Moreover,

$$V_2^{s(+)}(\underline{V}_1) = 0 \quad \text{and} \quad V_2^{s(-)}(\bar{V}_1) = -\infty,$$

where $V_2^{s(+)}$ denotes the right and $V_2^{s(-)}$ the left derivative. (ii) Under Assumption A.4, $V_2^s(V_1)$ is strictly concave if u_i is strictly concave, $i = 1, 2$, or over any interval such that $a^s(V_1)$ varies with V_1; under Assumption A.5, $V_2^s(V_1)$ is strictly concave over any interval such that $a^s(V_1)$ varies with V_1.

We use Assumption A.4 in Section 4 that considers the case where agents are risk averse. It requires two things: the first is essentially that the curvature of the deviation utility is less than the curvature of surplus as a function of actions. The second is that an optimal contract has $x_i > 0$, $i = 1, 2$ at every date. The latter follows, for example, for utility functions (such as those with constant relative risk aversion with coefficient of risk aversion greater than or equal to one) where $\lim_{x \to 0} u(x) = -\infty$. We use Assumption A.5 in Section 5 that considers the case where agents are risk neutral. It requires that the production function is more concave than the corresponding deviation utility. It is satisfied in many reasonable examples and Assumption A.5 is a generalized version of the condition given in Thomas and Worrall (1994).

In the one-sided action case where only one agent undertakes an action, it is known that the value function can *fail* to be differentiable (Thomas and Worrall 1994). It is perhaps surprising, then, that in this two-sided case we are able to establish differentiability. The key observation is that since optimal actions are positive, it is possible to vary both actions simultaneously, holding the future utilities constant, so as to vary V_1 whilst satisfying the self-enforcing and feasibility constraints.

3.3. Recursive formulation

We now use a recursive programming approach to examine optimal contracts. It is useful to work with net consumption x_i as a choice variable instead of consumption c_i. The Markov assumption on the evolution of states and the infinite time horizon, together with the observation that all the self-enforcing constraints are forward looking, means that the set of continuation utilities corresponding to a dynamic relational contract depends only on the state r and is independent of the past history. $V_2^s(V_1)$ is characterized as follows:

Proposition 3. $V_2^s(V_1)$, $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$, is a solution to the following program

[P1] $$V_2^s(V_1) = \max_{a \geq 0, x \geq \underline{x}, (V_1^r \in \mathbb{R})_{r \in \mathcal{S}}} \left\{ u_2(x_2) + \delta \sum\nolimits_{r \in \mathcal{S}} \pi_{sr} V_2^r(V_1^r) \right\}$$

subject to

(3.1a) $$u_1(x_1) + \delta \sum\nolimits_{r \in \mathcal{S}} \pi_{sr} V_1^r \geq V_1 : \qquad \lambda$$

(3.1b) $$V_1 \geq D_1^s(a_2) : \qquad \mu_1$$

(3.1c) $$u_2(x_2) + \delta \sum\nolimits_{r \in \mathcal{S}} \pi_{sr} V_2^r(V_1^r) \geq D_2^s(a_1) : \qquad \mu_2$$

(3.1d) $$V_1^r \geq \underline{V}_1^r : \qquad \delta \pi_{sr} \nu_1^r$$

(3.1e) $$V_1^r \leq \bar{V}_1^r : \qquad \delta \pi_{sr} \nu_2^r$$

(3.1f) $$x_i + a_i \geq 0 : \quad i, j = 1, 2, \quad i \neq j \qquad \gamma_i$$

(3.1g) $$x_1 + x_2 \leq z^s(a_1, a_2) : \qquad \psi$$

The non-negative Lagrangian multipliers are indicated after each inequality. The expected discounted utility V_1 of agent 1 (in state s) is the state variable in this programming problem. The value function $V_2^s(V_1)$ represents the Pareto-frontier of the set of dynamic relational contracts in the space of continuation utilities. It describes how the maximum continuation utility to agent 2 changes as the continuation utility of agent 1 is changed. The inequality (3.1a) is the *promise-keeping constraint* that requires that the contract delivers at least the current discounted utility. The inequalities (3.1b) and (3.1c) are the self-enforcing constraints corresponding to the inequalities given in (2.2). The constraints (3.1d) and (3.1e) reflect that the continuation utility for agent 1 in state r must lie in the interval $[\underline{V}_1^r, \bar{V}_1^r]$. Inequalities (3.1f) and (3.1g) are the feasibility constraints.

We denote a solution to [P1] by $(a^s(V_1), x^s(V_1))$ and continuation utilities $(V_1^{s,r}(V_1))$. It can be shown that $a^s(V_1)$ is unique; however, $x^s(V_1)$ and $V_1^{s,r}(V_1)$ need not be. Corresponding to this solution, and abusing notation, we define the surplus $z^s(V_1) := z^s(a_1^s(V_1), a_2^s(V_1))$. We discuss the properties of $z^s(V_1)$ below, but we refer to the maximal value of $z^s(V_1)$ for $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$ as the *constrained maximal surplus* and the actions that maximize this surplus as the *constrained surplus-maximizing* (CSM) actions. Let $\bar{a}(s)$ denote the CSM action in state s.[12] If the CSM actions are equal to the first-best actions $\bar{a}(s) = a^*(s)$ (and hence the constrained maximal surplus equals the first-best surplus), then we say that the first-best is *sustainable* in state s. We denote the set of states in which the first-best actions are sustainable as $\mathcal{S}_* \subseteq \mathcal{S}$ and denote its complement by $\mathcal{S}_*^c$ (it is possible that $\mathcal{S}_* = \emptyset$ or $\mathcal{S}_*^c = \emptyset$). A first-best allocation (FBA) will involve the first best actions, $a^*(s)$, in each state and date *and* complete risk-sharing (that is, net consumption $x^*(s)$ with $x_1^*(s) + x_2^*(s) = z^s(a^*(s))$ such that $u_2'(x_2^*(s))/u_1'(x_1^*(s))$ is constant over all states and dates).

An optimal contract is computed recursively. Start from some given initial value for agent 1's lifetime utility, $V_1(s_0)$ in state s_0. The solution to the programming problem provides optimal values for $a(s_0)$ and $x(s_0)$ in state s_0 by setting $V_1 = V_1(s_0)$ in [P1]. The solution also determines the continuation utilities for $V_1^{s_0,r}(V_1(s_0))$ in each possible subsequent state r. At date $t = 1$ and history $s^1 = (s_0, s_1)$, the value for V_1 is determined by the solution for the continuation utility at date $t = 0$ for the appropriate state and the solution to the date $t = 1$ programme determines $a(s^1)$ and $x(s^1)$. The process is repeated to determine $\{a(s^t), x(s^t)\}_{t=0}^{\infty}$. Doing this for each $V_1(s_0) \in [\underline{V}_1^{s_0}, \bar{V}_1^{s_0}]$ determines the set of optimal contracts.

3.4. First-order conditions

From Proposition 2 the Pareto-frontier is continuously differentiable and the range of absolute slopes of the frontier is $\mathbb{R}_+ \cup \{\infty\}$. Let $\sigma_s(V_1) := -V_2^{s\prime}(V_1)$ and $\sigma_{s,r}^+(V_1) := -V_2^{r\prime}(V_1^{s,r}(V_1))$ be the (absolute) slopes of the Pareto-frontiers, where $\sigma_s \colon [\underline{V}_1^s, \bar{V}_1^s] \to \mathbb{R}_+ \cup \{\infty\}$ is strictly increasing since the Pareto-frontier is strictly concave. The envelope condition for [P1] is $-\sigma_s(V_1) = -\lambda + \mu_1$. Using this condition, differentiating with respect to x_i,

[12] In principle, there may be dynamic relational contracts in which there are actions that achieve a higher surplus but at the cost of lower future surplus. Our definition considers only optimal contracts. However, in both the risk-neutral and risk-averse cases that we consider below, the two concepts coincide and the CSM actions do maximize $z^s(d_1, d_2)$ subject to the self-enforcing constraints. It will also be shown below that in the cases we consider, the CSM actions are unique.

a_i and V^r in [P1], and rearranging gives the first-order conditions:

$$\sigma^+_{s,r}(V_1) - \sigma_s(V_1) = -\sigma_s(V_1)\frac{\mu_2}{1+\mu_2} + \frac{\mu_1}{1+\mu_2} + \frac{\nu^r_1 - \nu^r_2}{1+\mu_2} \tag{3.2a}$$

$$\sigma^+_{s,r}(V_1) = \frac{u_2'(\cdot)}{u_1'(\cdot)} + \frac{\gamma_2 - \gamma_1}{u_1'(\cdot)(1+\mu_2)} + \frac{\nu^r_1 - \nu^r_2}{1+\mu_2} \tag{3.2b}$$

$$\frac{\mu_j}{1+\mu_2}\frac{\mathrm{d}D^s_j}{\mathrm{d}a_i} = \frac{\partial z^s}{\partial a_i}\left(u_2'(\cdot) + \frac{\gamma_2}{1+\mu_2}\right) + \frac{\gamma_i}{1+\mu_2} \quad i,j = 1,2 \text{ and } i \neq j. \tag{3.2c}$$

Since the range of absolute slopes of the frontier is $\mathbb{R}_+ \cup \{\infty\}$, it is intuitive that $\sigma^+_{s,r}(V_1)$ is the same for each future state $r \in \mathcal{S}$. To see this first suppose that $\nu^r_1 > 0$. In this case $V^{s,r}_1(V_1) = \underline{V}^r_1$, $\sigma^+_{s,r}(V_1) = 0$ and by a complementary slackness condition $\nu^r_2 = 0$. Then using equation (3.2a), $-\sigma_s(V_1) - \mu_1 = \nu^r_1 > 0$ which gives a contradiction since $\sigma_s(V_1)$ and μ_1 are non-negative. A similar argument can be made to show that $\nu^r_2 = 0$. Since $\nu^r_i = 0$, it follows from (3.2a) that $\sigma^+_{s,r}(V_1)$ is independent of r and we write $\sigma^+_s(V_1)$ for this common future value. This property greatly simplifies the dynamics of the contracting problem.

It follows directly from the first-order conditions (3.2c) that in an optimal contract (i) there is only ever underinvestment, $a_i(s^t) < a^*_i(a_j(s^t), s)$, if at least one of the agents is constrained; and (ii) if agent i has positive consumption, then he/she does not overinvest, $a_i(s^t) \leq a^*_i(a_j(s^t), s)$. To see the intuition for the first part, suppose that agent 1 is unconstrained. If agent 2 were underinvesting, he could increase investment and generate more surplus. The surplus would be enough to compensate him for the extra investment and agent 1 won't default because she is unconstrained. Thus, it would be possible to find a better contract, yielding a contradiction. Similarly, to see the second part, suppose that agent 1 is overinvesting. Then she could reduce her investment. This relaxes agent 2's self-enforcing constraint (keeping consumptions now and future promises the same). However, output has fallen, so aggregate consumption must fall. If agent 1 has positive consumption, it is possible to keep the consumption of agent 2 the same, while the utility of agent 1 increases because she has cut her investment from above the conditionally efficient level.

There is also a simple corollary to these results: a) both agents cannot be overinvesting (because one agent must have positive consumption); b) an agent cannot be permanently overinvesting because consumption must be positive at some future date – otherwise the self-enforcing constraint would not be satisfied.

4. Risk aversion

For this section we assume that agents are risk averse: we strengthen Assumption 1 and assume that u_i is strictly concave for $i = 1, 2$, and use Assumption A.4 from the Appendix. In particular, it is assumed that net consumption and hence, consumption is strictly positive in an optimal contract. It will follow from this that overinvestment is not a feature of an optimal contract. The allocation of net consumption between agents may vary, potentially considerably, across states even in the long-run. Thus, it is important to examine how allowing for risk aversion affects optimal contracts.

4.1. Characterization of optimal contracts

In this sub-section, we consider some properties of the optimal contract and surplus as V_1 varies in a given state, and how the contract is updated period-by-period: in particular, how the ratio of marginal utilities changes from one period to the next. In the following sub-section, we consider the long-run properties of the optimal contract showing that it evolves towards a stationary distribution and study when this stationary distribution does or does not depend on the value of agent 1's lifetime utility $V_1(s_0)$.

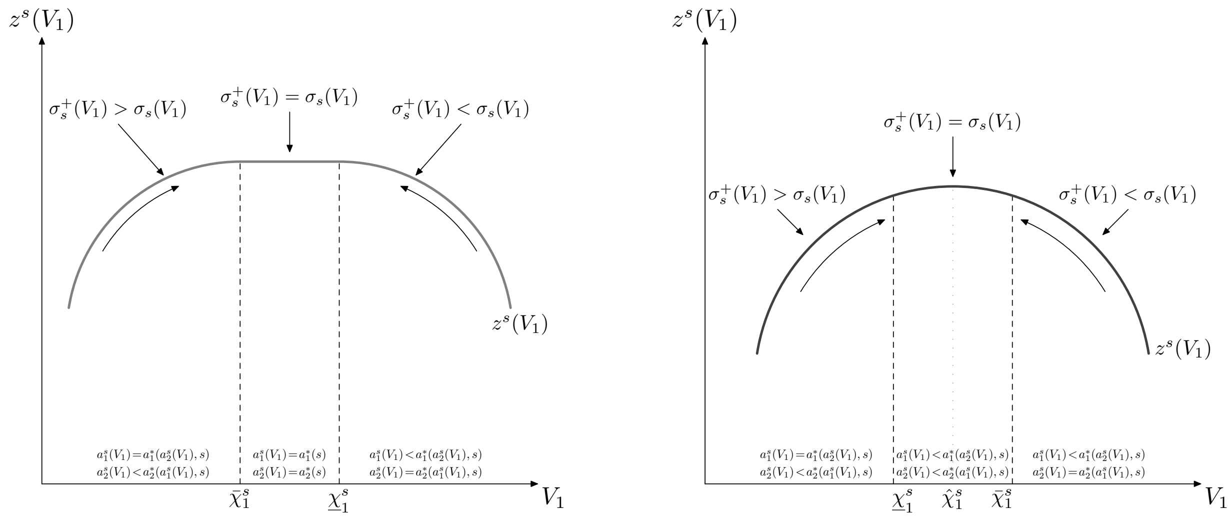
Proposition 4. With risk-averse agents and under Assumption A.4 (i) there is no overinvestment, $\partial z^s(a(s^t))/\partial a_i \geq 0$, $i = 1, 2$, all s^t; (ii) surplus $z^s(V_1)$ is a concave differentiable function (strictly concave if $s \in \mathcal{S}_*^c$) with maximum at unique CSM actions; (iii) at V_1 such that $z^s(V_1)$ is maximized, if $s \in \mathcal{S}_*^c$ both constraints bind and $a^s(V_1) < a^*(s)$, and if $s \in \mathcal{S}_*$ efficient actions $a^*(s)$ are sustainable by definition; (iv) for each $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$, $\sigma_s^+(V_1)$, the (absolute value of the) common slope of the Pareto-frontiers next period, and $\sigma_s(V_1)$, the slope of the current Pareto-frontier, satisfy

$$\sigma_s^+(V_1) - \sigma_s(V_1) = u_2'\left(x_2^s(V_1)\right) \frac{\mathrm{d}z^s(V_1)}{\mathrm{d}V_1}. \tag{4.1}$$

The intuition for (i) was discussed above in Section 3.4 when $c_i > 0$ for $i = 1, 2$. Properties (ii) and (iii) are illustrated in Figure 1.[13] Equation (4.1) in part (iv) is fundamental to understanding the dynamics of an optimal contract. It is easy to interpret. Consider a (small) unit increase in V_1. The effect on agent 2's discounted utility is to change it by approximately $V_2^{s\prime}(V_1) = -\sigma_s(V_1)$ units. One way to effect this change (as good as any other at the optimum) is to hold the current utility of agent 1 constant (giving any change in the current surplus to agent 2) and increase V_1^r in each state r, the next-period continuation utilities of agent 1, by $1/\delta$. The effect on agent 2's current utility is $u_2'(x_2^s(V_1)) \times (\mathrm{d}z^s(V_1)/\mathrm{d}V_1)$. The effect on the discounted continuation utility of agent 2 is to decrease it by $\sigma_s^+(V_1)$, the same for all future states. The combined effect for agent 2 is $u_2'(x_2^s(V_1)) \times (\mathrm{d}z^s(V_1)/\mathrm{d}V_1) - \sigma_s^+(V_1)$. Since the overall change in utility for agent 2 is $-\sigma_s(V_1)$, we can equate to get equation (4.1).

The implication for the dynamics of optimal contracts is illustrated in Figure 1. Consider starting from a value of V_1 below the level that maximizes surplus. In this region, agent 1's constraint binds ($D_1^s(a_2^s(V_1)) = V_1$) and a_2 is kept inefficiently low ($a_2^s(V_1) < a_2^*(a_1^s(V_1), s)$) to prevent agent 1 from deviating. In this region, V_2 may be high enough to allow a_1 to be conditionally efficient ($a_1^s(V_1) = a_1^*(a_2^s(V_1), s)$) without violating agent 2's constraint, but if $s \in \mathcal{S}_*^c$, then, closer to the surplus maximizing value of V_1, both constraints will bind and a_1 will be inefficiently low. Also, in this region, $\mathrm{d}z^s(V_1)/\mathrm{d}V_1 > 0$, so equation (4.1) implies that $\sigma_s^+(V_1) > \sigma_s(V_1)$. In particular, if there is a single state or if the same state recurs,

[13] Where $\bar{\chi}_1^s$ and $\underline{\chi}_1^s$ are the values for V_1 such that agent 1's constraint binds for $V_1 \leq \bar{\chi}_1^s$, while agent 2's constraint binds for $V_1 \geq \underline{\chi}_1^s$; surplus is maximized at $\hat{\chi}_1^s$ in case (b). See the Supplementary Material for further details. Note that both constraints bind for values of $V_1 \in (\underline{\chi}_1^s, \bar{\chi}_1^s)$ in Figure 1b. This contrasts with pure risk-sharing models with limited commitment, for example, Kocherlakota (1996) or Thomas and Worrall (1988), where at most one self-enforcing constraint binds at any one time in any non-trivial optimum.



(a). $s \in \mathcal{S}_*$ (b). $s \in \mathcal{S}_*^c$

Figure 1: Surplus Function $z^s(V_1)$

the change in V_1 is as indicated by the arrows in Figure 1. In this case, surplus will be higher next period as the increase in V_1 allows the extent of agent 2's underinvestment to be reduced, and by enough to offset any increase in underinvestment by agent 1. (We discuss the implications when states switch below.) A symmetric argument applies to the dynamics for high values of V_1.

4.2. Long-run dynamics

To examine long-run convergence, we treat choices at date t as random variables and write $x_1(t)$ for the random value of net consumption of agent 1 at date t after history s^t etc. Define $\rho(t) := u_2'(x_2(t))/u_1'(x_1(t))$ to be the ratio of marginal utilities at date t ($\rho(t) = \sigma(t+1)$). In this subsection we focus on the long-run properties of $\rho(t)$.

With more than one state, convergence to constrained surplus maximization may not occur because there is a conflict between risk sharing and surplus maximization. To achieve surplus maximization in state s, the distribution of consumption may differ from that in $s' \neq s$ and therefore, an optimal contract must (dynamically) trade-off risk sharing against surplus maximization.

As already described, there is a (possibly trivial) interval of marginal utility ratios corresponding to maximum surplus in any state s. Let $[\underline{\rho}_s, \bar{\rho}_s]$ denote this interval in state s.[14] By equation (4.1), the marginal utility ratio is unchanged from the previous period if (and only if) surplus is maximized today (i.e., $\rho(t) \in [\underline{\rho}_{s_t}, \bar{\rho}_{s_t}]$). Thus, a constant marginal utility ratio requires that $\Omega := \cap_{s\in\mathcal{S}}[\underline{\rho}_s, \bar{\rho}_s] \neq \emptyset$. The set Ω is non-empty when an FBA is sustainable, in which case the ratio is constant. If Ω is not only non-empty but a non-trivial interval, then there are multiple FBAs. Moreover, if an FBA is sustainable, then monotone

[14]For $s \in \mathcal{S}_*$, $\underline{\rho}_s = \sigma_s(\bar{\chi}_1^s)$ and $\bar{\rho}_s = \sigma_s(\underline{\chi}_1^s)$ and for $s \in \mathcal{S}_*^c$, $\underline{\rho}_s = \bar{\rho}_s = \sigma_s(\hat{\chi}_1^s)$ (see Figure 1).

convergence to an FBA occurs. If however, Ω is empty, or if CSM actions are not always efficient, an FBA is not sustainable and the marginal utility ratio may not converge to a single value. Nevertheless, under a weak regularity condition, it does converge to a unique long-run invariant distribution, independent of the initial conditions.

To describe the evolution of the marginal utility ratio, let $F_t^{(V_1(s_0))}\colon \mathbb{R}_+ \to [0,1]$ denote the distribution function of $\rho(t)$ at date t given the initial value $V_1(s_0)$. This leads us to the following general convergence theorem.[15]

Theorem 1. a) Suppose an FBA is sustainable. Then an optimal contract converges with probability one to an FBA: $||a(t) - a^*(s_t)|| \to 0$ and the random sequence $\{\rho(t)\}$ is (weakly) monotone, with probability one. If there exist multiple FBAs, then the limit FBA depends upon $V_1(s_0)$.
(b) Suppose instead that an FBA is not sustainable. Then, provided $\pi_{ss} > 0$ for all s, $F_t^{(V_1(s_0))}$ converges weakly to a unique distribution independent of $V_1(s_0)$. Either (i) this distribution is degenerate, in which case dynamics are as in part (a), with stationary limit contract with CSM actions $\bar{a}(s)$ in each state, or otherwise (ii) this distribution is non-degenerate, and current surplus is not maximized in the long run: $||a(t) - \bar{a}(s_t)|| \to 0$ with probability zero.

In part (a) of Theorem 1, there is convergence to an FBA. There is a (possibly trivial) interval of the ratio of marginal utilities given by the set Ω that are compatible with efficient actions and a constant marginal utility ratio. Convergence will be to the lower endpoint of Ω if the initial marginal utility ratio is below the interval; to the upper endpoint of Ω if initial marginal utility ratio is above the interval; and the sequence of marginal utility ratios will be constant if the initial marginal utility ratio belongs to Ω. The dynamics are similar in Part (b)(i), which considers the case where there is a marginal utility ratio consistent with CSM actions in each state. Convergence is to the CSM actions and to this (unique marginal utility) ratio. This case arises if there is a single state but CSM actions are not efficient.[16] If there are multiple states and CSM actions in each state are inefficient, then this case is possible but not generic in the sense that a small perturbation of either ϕ_i^s or y^s in any state s will lead to the case of Part (b)(ii).

Part (b)(ii) of Theorem 1 provides a description of what happens when there is a conflict between surplus maximization and risk sharing. The optimal contract exhibits a second-best property. The marginal utility ratio $\rho(t)$ does not settle down to a single value, and whenever it differs across two dates $t-1$ and t, actions at date t will not be CSM.[17] By contrast, in the risk-neutral case, we show that once the stationary phase is reached surplus is maximized in each state by varying the continuation utility to allow the constrained maximal surplus to be achieved (Theorem 2). For example, if the state changes from one in

[15] We use $||\cdot||$ to denote the Euclidean norm.

[16] For the single state case irrespective of whether $s \in \mathcal{S}_*$ or not, $\{\rho(t)\}$ monotone implies $\{z^s(t)\}$ is monotone increasing, converging to constrained maximal surplus, as indicated by the arrows in Figure 1.

[17] Formally, $\rho(t{-}1) \neq \rho(t)$ corresponds to $\sigma(t) \neq \sigma(t{+}1)$, and thus, from (4.1), $\mathrm{d}z^{s_t}(V_1(t))/\mathrm{d}V_1 \neq 0$. Hence, actions at date t are not CSM, as claimed.

which agent 1 can claim most of output to one in which roles are reversed, sufficient surplus and future utility is reallocated to agent 2 to satisfy his self-enforcing constraint at the CSM actions for that state. However, in the risk-averse setting of part (b)(ii) of Theorem 1, risk-sharing considerations make such an immediate step change undesirable. It is better to hold agent 1's action at the later date inefficiently low, keeping agent 2's default payoff from rising too much, thereby relaxes the latter's self-enforcing constraint meaning that the share going to agent 2 does not rise to that consistent with the CSM actions.

To better understand this dynamic trade-off between surplus maximization and risk sharing suppose to the contrary that the ratio of marginal utilities differs across two dates $t-1$ and t, but actions at date t *are* CSM. Then a simple change in the contract at $t-1$ and t can produce a Pareto-improvement. Consider the case where $\rho(t-1) > \rho(t)$. Initially hold actions fixed at both dates and increase $x_1(t)$ by a small amount, but reduce $x_1(t–1)$ to leave $V_1(t–1)$ unchanged. If surplus were unchanged at t, this would improve risk sharing and lead to a Pareto-improvement because $V_2(t–1)$ would increase. However, because $x_2(t)$, and hence $V_2(t)$, have fallen, agent 2's self-enforcing constraint may be violated at the initial actions (and will be, if the CSM actions are below the first-best). In order not to violate agent 2's self-enforcing constraint, agent 1's action at date t can be reduced. Correspondingly, agent 2's action can be increased because $V_1(t)$ has risen. Critically, although this change may reduce surplus at date t, it does so only by a second-order amount since, by assumption, the original actions at date t were CSM.[18] Consequently, a Pareto-improvement results, contradicting the supposed optimality of the original situation.

4.3. Pure risk-sharing

We now compare our results to the standard limited commitment, two-agent, pure risk-sharing model of Thomas and Worrall (1988), Kocherlakota (1996), Ligon et al. (2002). To do this, for simplicity we consider a special case of our hold-up model with additive production, $y^s(a) = f_1^s(a_1) + f_2^s(a_2)$, and proportional defaults, $\phi_i^s(a) = \theta_{i1}^s f_1^s(a_1) + \theta_{i2}^s f_2^s(a_2)$ where $\theta_{ij}^s \geq 0$, $i,j = 1,2$, and $\sum_{i=1}^2 \theta_{ij}^s = 1$, $j = 1,2$. Our hold-up assumption requires $\theta_{ij}^s > 0$, $i,j = 1,2$, $i \neq j$, all s. Holding technology and preferences fixed, consider the limit case *where hold-up vanishes*: $\theta_{ij}^s = 0$, $i,j = 1,2$, $i \neq j$, all s. This corresponds to the pure-risk sharing model. In any optimal contract of this limit model actions are clearly efficient, as are actions in the breakdown, so only efficient levels play any role. Agent i's "endowment" in state s is $f_i^s(a_i^*(s)) - a_i^*(s)$ and breakdown utility is $u(f_i^s(a_i^*(s)) - a_i^*(s))$.

We establish that the dynamics of the hold-up model converge to that of the risk-sharing model. In the latter, as is well known, dynamics are summarized in a simple updating rule for $\rho(t)$ (which fixes surplus division given surplus depends only on s). We characterize how the corresponding updating rule in the hold-up model converges to the risk-sharing one as hold-up disappears. One application of this is that it allows us to characterize general properties of the hold-up dynamics for cases where hold-up is low.

[18] The change in surplus would be second order when V_1 and V_2 are varied *according to the Pareto frontier* at t starting from maximum surplus; because the frontier's slope is $-\rho(t)$ at maximum surplus, the change we construct also only has a second-order effect. Also, note that, by construction, the self-enforcing constraints hold at t, and since $V_1(t-1)$ is unchanged and $V_2(t-1)$ is increased, they also hold at $t-1$.

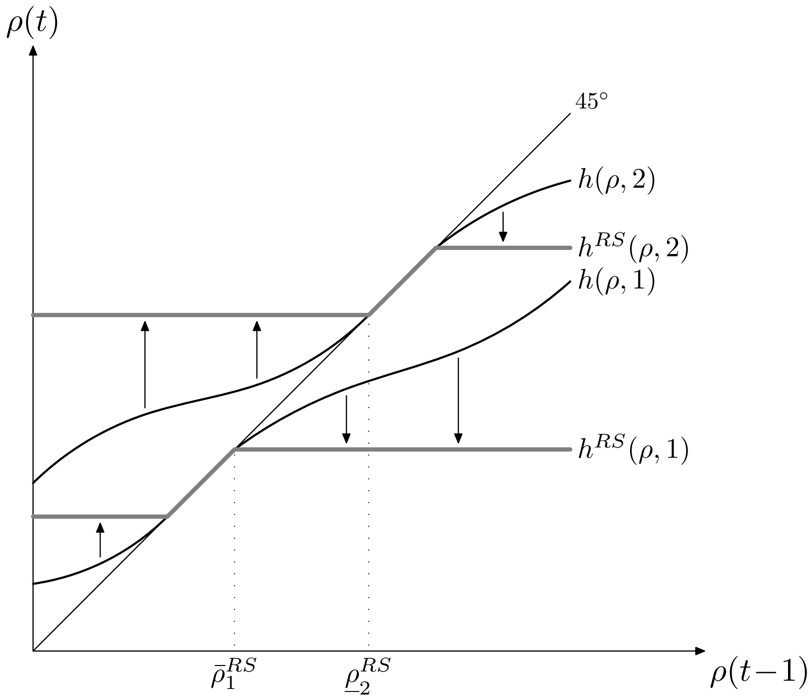


Figure 2: Convergence to Pure Risk-Sharing

From Ligon et al. (2002), the updating rule in the pure risk-sharing case, which we write $\rho(t) = h^{RS}(\rho(t-1), s_t)$, has the property that there is a (possibly degenerate) interval $[\underline{\rho}_s^{RS}, \bar{\rho}_s^{RS}]$ for each s such that $h^{RS}(\rho(t-1), s) = \bar{\rho}_s^{RS}$ if $\rho(t-1) > \bar{\rho}_s^{RS}$; $h^{RS}(\rho(t-1), s) = \rho(t-1)$ if $\rho(t-1) \in [\underline{\rho}_s^{RS}, \bar{\rho}_s^{RS}]$ and $h^{RS}(\rho(t-1), s) = \underline{\rho}_s^{RS}$ if $\rho(t-1) < \underline{\rho}_s^{RS}$. Moreover, whenever optimal contracts that improve on autarky exist (if there is more than one distinct state, and δ is close enough to 1), each $[\underline{\rho}_s^{RS}, \bar{\rho}_s^{RS}]$ is non-degenerate (Proposition 2(iv) in Ligon et al. 2002).

Likewise, in the hold-up model we can also use $(\rho(t-1), s_t)$ as the state variable. (By $\rho(t-1) = \sigma(t)$, this is equivalent to $(\sigma(t), s_t)$.) Thus, the evolution of the contract can be represented by $\rho(t) = h(\rho(t-1), s_t)$, where $h\colon \mathbb{R}_+ \cup \{\infty\} \times \mathcal{S} \to \mathbb{R}_+$ (see the Appendix for details and characterization). The updating functions $h(\rho, s)$ converge to those of the pure risk-sharing model as the hold-up problem diminishes. Moreover, for $\rho(t-1)$ within the interior of the interval $[\underline{\rho}_{s_t}^{RS}, \bar{\rho}_{s_t}^{RS}]$, when hold-up is small enough, optimal actions at t are at the first-best levels and so $\rho(t) = \rho(t-1)$. An illustration of this convergence for two states is depicted in Figure 2.[19]

Proposition 5. For each state $s \in \mathcal{S}$, (i) for all $\rho \in \mathbb{R}_+$, $h(\rho, s) \to h^{RS}(\rho, s)$ as $\theta_{ij} \to 0$, $i, j = 1, 2$, $i \neq j$, all s. (ii) If optimal contracts in the risk-sharing problem improve upon

[19]Figure 2 is drawn with $h^{RS}(\rho, s)$ and $h(\rho, s)$ coinciding along the 45° line. This is for illustration only. Close to the limit as hold-up vanishes it is approximately true that the lines $h^{RS}(\rho, s)$ and $h(\rho, s)$ are coincident along the 45° line, except near the end-points of the segment of $h^{RS}(\rho, s)$ that lies on the 45° line. In general, we have no result on how the intersections of $h^{RS}(\rho, s)$ and $h(\rho, s)$ with the 45° line are related.

autarky, then for any η satisfying $(1/2)(\bar{\rho}_s^{RS} - \underline{\rho}_s^{RS}) > \eta > 0$, all s, there exists $\epsilon > 0$ such that for $\theta_{ij}^s < \epsilon$, $i, j = 1, 2$, $i \neq j$, all s, $h(\rho, s) = \rho$ for all $\rho \in [\underline{\rho}_s^{RS} + \eta, \bar{\rho}_s^{RS} - \eta]$.

One well-known feature of the pure risk sharing model is the "amnesia" property that once one of the agents is constrained, then the previous history is irrelevant to the future evolution of the optimal contract. This property no longer applies in our model of risk averse agents with actions. Suppose that agent 2's self-enforcing constraint binds at date t. In the risk-sharing problem, this fixes his continuation utility and there is a unique optimal way of delivering this continuation utility independently of past history and, in particular, independently of the previous ratio of marginal utilities. This can be seen in the flat sections of the functions $h^{RS}(\rho, s)$ in Figure 2. In the hold-up problem, by contrast, agent 2's self-enforcing constraint can be relaxed by cutting agent 1's action. Although this change may reduce surplus, sacrificing surplus can be offset by improved risk sharing and the incentive to do this will vary with the lagged marginal utility ratio. The logic of trading off surplus to improve risk sharing is similar to the explanation given above for why the partial insurance case involves optimal actions that are not CSM, even in the long run. This result is illustrated in Figure 2 by the fact that the functions $h(\rho, s)$ are upward sloping even away from the 45° line. Thus, even when an agent is constrained, past history affects the current actions and consumption and the future evolution of the optimal contract. The amnesia property fails.

5. Risk neutrality

For this section, we use Assumption A.5 and suppose that both agents are risk neutral, in particular, that $u_i(x) = x$ and that $\underline{x}_i = -\infty$ for $i = 1, 2$. In this case, the non-negativity constraint on consumption (limited liability) plays a key role. We show that an optimal contract exhibits a two-stage property. It starts with a backloading phase in which one of the agents consumes all of the output. This agent never overinvests, while the other agent overinvests. The second phase is stationary and actions are CSM. Therefore, if $s \in \mathcal{S}_*$, actions are at the first-best for both agents. If $s \in \mathcal{S}_*^c$, both agents underinvest and have positive consumption. Depending on the initial division of surplus however, the optimal contract might start off in the stationary phase in which case the first backloading phase does not exist.

The lower bound for the deviation utility is strictly positive. Therefore, the Pareto-frontier is defined on $\Lambda^s := [\underline{V}_1^s, \bar{V}_1^s] \subset \mathbb{R}_{++}$. It can be shown that the frontier is strictly concave if at least one of the self-enforcing constraints is binding. If V_1 is in an interval where the efficient actions are sustainable (such values may not exist), then the frontier is linear with slope of -1 in this interval. In either case, CSM actions are unique.

Consider three (not necessarily disjoint) subsets of Λ^s: $A^s = \{V_1 \in \Lambda^s \colon c_1^o = 0\}$, $B^s = \{V_1 \in \Lambda^s \colon c_1^o > 0 \text{ and } c_2^o > 0\}$ and $C^s = \{V_1 \in \Lambda^s \colon c_2^o = 0\}$ where (c_1^o, c_2^o) represents an optimal value for consumption at V_1. Note that $A^s \cup B^s \cup C^s = \Lambda^s$. Also note that A^s can be non-empty and C^s empty or vice-versa (examples of this type can be constructed). We know from our previous discussion that if agent 1 overinvests, this can only occur for $V_1 \in A^s$, and if agent 2 overinvests, this occurs for $V_1 \in C^s$. Also, since optimal actions are positive, output and aggregate consumption is positive, and consequently, it is not possible that both $\gamma_i > 0$ for the same V_1. Equally, for $V_1 \in A^s$, $c_2 > 0$, and hence, the multiplier $\gamma_2 = 0$.[20] We also know from Proposition 1 that if $c_1 = 0$, and therefore, that agent 2 gets all the consumption, then agent 2 is unconstrained, and hence, $\mu_2 = 0$. Likewise, for $V_1 \in C^s$, $\gamma_1 = \mu_1 = 0$. Consumption for both agents is positive for $V_1 \in B^s$, so that $\gamma_1 = \gamma_2 = 0$.

Consider the subset A^s. Using $\gamma_2 = \mu_2 = 0$, we have from the first-order conditions (3.2a) - (3.2c) that:

$$\sigma_s^+(V_1) = 1 - \gamma_1 = \frac{\partial y^s(a_1, a_2)}{\partial a_1}, \tag{5.1a}$$

$$\sigma_s(V_1) = 1 - \gamma_1 - \mu_1 = \frac{\partial y^s(a_1, a_2)}{\partial a_1} - \frac{\partial z^s(a_1, a_2)}{\partial a_2} \left(\frac{\mathrm{d}D_1^s}{\mathrm{d}a_2} \right)^{-1}. \tag{5.1b}$$

Hence, for $V_1 \in A^s$, $1 \geq \sigma_s^+(V_1) \geq \sigma_s(V_1)$. From equation (5.1a) it follows that if $\sigma_{s,r}^+(V_1) < 1$, then $\gamma_1 > 0$, and $\partial y^s(a_1, a_2)/\partial a_1 < 1$, so that agent 1 is overinvesting. From equation (5.1b) it follows that agent 2 doesn't overinvest and may underinvest. A similar set of conditions apply for $V_1 \in C^s$ and imply $1 \leq \sigma_s^+(V_1) \leq \sigma_s(V_1)$ so that agent 1 doesn't overinvest and if $\sigma_s^+(V_1) > 1$, then agent 2 overinvests. For $V_1 \in B^s$, the first-order conditions show that $\sigma_s^+(V_1) = 1$, so there is no overinvestment. As a measure of the extent of overinvestment let $\zeta_i^s := \max\{0, -ln(\partial y^s(a_1, a_2)/\partial a_1)\}$ and $\zeta^s := \max\{\zeta_1^s, \zeta_2^s\}$. Hence, $\zeta^s > 0$ if there is overinvestment and is a measure of the distortion of the marginal product below the efficient level.[21]

We now state our two-phase characterization theorem. Here, for convenience, we also treat contracts as sequences of random variables, writing $a_i(t)$ rather than $a_i(s^t)$ etc.

Theorem 2. In an optimal contract, there is a random time $\hat{t}$, $0 \leq \hat{t} < \infty$ with probability one, such that:

Stationary phase ($t \geq \hat{t}$): Optimal actions maximize the surplus $z^s(a_1, a_2)$ subject to the self-enforcing constraints, and hence, are CSM. The optimal actions depend only on the state s_t and are therefore independent of the initial conditions. There is no overinvestment: $a_i(t) \leq a_i^*(a_j(t), s_t)$ for $i, j = 1, 2$, $i \neq j$. For $s_t \in \mathcal{S}_*$, therefore, optimal actions and the corresponding surplus are first best: $a(t) = a^*(s_t)$ and $z^{s_t}(a(t)) = z^{s_t}(a^*(s_t))$. For $s_t \in \mathcal{S}_*^c$, the self-enforcing constraints bind for both agents, $c_i > 0$ for $i = 1, 2$, and there is underinvestment: $a_i(t) < a_i^*(a_j(t), s_t) \leq a_i^*(s_t)$ for $i, j = 1, 2$, $i \neq j$.

Backloading phase ($t < \hat{t}$): Overinvestment declines during the backloading phase: in particular, $\zeta(t)$ is weakly decreasing with $\zeta(\hat{t} - 1) = 0$. Backloading only applies to one agent, i, whose identity depends on the initial surplus split: this agent overinvests and has zero consumption at each $t < \hat{t} - 1$. In the final period of backloading, at date $\hat{t} - 1$, there is no overinvestment: $a_i(\hat{t} - 1) \leq a_i^*(s_{\hat{t}-1})$, but $a_j(\hat{t} - 1) < a_j^*(s_{\hat{t}-1})$ for $j \neq i$. Moreover, if at any two dates t and $t' > t$ the same state s occurs, then underinvestment diminishes and surplus

[20] Since the multiplier is unique, the conclusion that $\gamma_2 = 0$ is valid even if V_1 also belongs to B^s or to C^s. The same argument can be made for the other subsets and multipliers.

[21] In subset A^s, $\zeta^s = -\ln \sigma_s^+(V_1)$ and in subset C^s, $\zeta^s = \ln \sigma_s^+(V_1)$.

increases: $\partial y^s(a(t))/\partial a_j \geq \partial y^s(a(t'))/\partial a_j \geq 1$ for $\hat{t}-1 \geq t' > t$ and $z^s(a(t')) \geq z^s(a(t))$ for $\hat{t} \geq t' > t$.

For a given value of agent 1's lifetime utility $V_1(s_0)$, there corresponds a value σ_0. From Theorem 2, we can describe a typical path as follows. Suppose $\sigma_0 < 1$ (a symmetric argument applies if $\sigma_0 > 1$). Then one of two possible scenarios applies. Either $V_1(s_0) \in B^{s_0}$ or $V_1(s_0) \in A^{s_0}$. In the former case, $\hat{t} = 1$ and the contract moves to the stationary phase in each state at the next period. There is no overinvestment in this case. In the latter case, either $\zeta_1(0) = 0$ and $\hat{t} = 1$ as in the previous case, or $\zeta_1(0) > 0$ in which case $\hat{t} > 1$ and there is a backloading phase in which $c_1(t) = 0$ and agent 1 overinvests. Correspondingly, V_1 is sufficiently low that agent 1's self-enforcing constraint binds and agent 2 underinvests to avoid violating agent 1's self-enforcing constraint; by contrast V_2 is high enough that agent 2's self-enforcing constraint is slack.[22] The basic intuition for the backloading result is familiar from other dynamic contracting models. The claim is that if agent 2 is unconstrained and underinvesting, then agent 1 has zero consumption at all previous dates, her payments are optimally backloaded into the future. The idea is that if agent 1 has positive consumption, then backloading her consumption allows her later constraints to be relaxed, which in turn means agent 2 can increase his future investment level without violating agent 1's constraint. Since agents are risk neutral they do not care about the timing of consumption flows (keeping the action plans fixed) if the expected discounted value is the same, but the backloading will permit future surplus to be increased, leading to a Pareto-improvement. Consumption is backloaded to the maximum extent possible, $c_1(t) = 0$ throughout the phase, allowing maximum surplus to be achieved as quickly as possible. Furthermore, by increasing $a_1(t)$ above $a_1^*(a_2(t), s)$, with the extra output being allocated to agent 2, additional backloading can be achieved, and for a small amount of overinvestment, the reduction in surplus is second-order.[23] Two novel results in the two-sided environment concerning the backloading phase are the over-investment by the agent whose utility is backloaded (although over-investment does not persist into the stationary phase), and the fact that despite the possibility that property rights might vary radically and persistently between states, only one of the agents will ever be subject to backloading.

That there is overinvestment in the backloading phase is perhaps surprising given the hold-up problem and given that the literature, mentioned in the Introduction, that considers the case where only one agent takes an action finds that there is never any overinvestment. Consider the one-sided case with only agent 1 taking an action. If agent 2 gets sufficient of the surplus to allow a_1 to be more than a_1^* without agent 2 wanting to deviate, then the optimal contract will be stationary with $a_1 = a_1^*$. The benefit from overinvestment is that

[22] This characterization applies so long as $\zeta_1(t) > 0$ and assuming agent 1's self-enforcing constraint binds with a positive multiplier. With more than one state, we cannot rule out the possibility that in some states deviation utilities are so low that the self-enforcing constraints may not bind even when $\sigma(t) < 1$. In this latter case, from (3.2c) and (3.2a), $a_2(t) = a_2^*(a_1(t), s)$ and $\sigma^+(t) = \sigma(t)$.

[23] The incentive to overinvest diminishes over time (as can be seen from (5.1a), $\sigma^+(t)$ approaches 1). Equally, if the same state recurs along the path, underinvestment diminishes as the self-enforcing constraint is relaxed. The combined effect is that surplus $z^s(a(t))$ increases, and reaches a maximum when $\sigma(t) = 1$.

it allows more backloading of agent 1's utility when $c_1 = 0$. In this case however there is no benefit, but an efficiency cost, *and* backloading can only increase agent 2's incentive to renege in the future, potentially necessitating lower (inefficient) future actions by agent 1. Thus, there is no overinvestment.

6. Conclusion

In this paper, we have analyzed the dynamic properties of a relational contract between two agents both of whom undertake a costly investment or action that yields joint benefits. We have shown that optimal contracts exhibit different properties depending on whether agents are risk neutral or risk averse. In the risk-neutral case, actions may be either above or below the efficient level and actions and the division of the surplus converge monotonically to a stationary solution at which actions are constrained surplus maximizing (either both are first-best or both are below the first-best level). In the risk-averse case, we also establish a convergence result but convergence may or may not be monotonic depending on whether it is possible to sustain a first-best allocation or not. We have demonstrated that the optimal contract converges to the pure-risk sharing results of Kocherlakota (1996) as our hold-up problem vanishes.

In the risk-averse case there is an interesting trade-off between hold-up and risk-sharing. The hold-up problem creates an opportunity to relax the default constraint by lowering actions. This in turn allows more risk-sharing to be achieved without leading to default. It would be interesting to evaluate whether the gain in risk-sharing would ever be sufficient to offset the loss in surplus created by the original hold-up problem. This is a difficult question because without additional structure to the model little can be said about the long run distribution of the optimal contract.

Appendix

Statements of lemmas for Section 2

Lemma 1. Under Assumption 2, for $i, j = 1, 2,\ i \neq j$ and for each $s \in \mathcal{S}$, the conditionally efficient action, $a_i^*(a_j, s)$, is single-valued, weakly increasing and continuous in a_j.

Lemma 2. Under Assumption 3, for $i, j = 1, 2,\ i \neq j$ and for each $s \in \mathcal{S}$, the Nash best-response, $a_i^N(a_j, s)$, is single-valued, weakly increasing and continuous in a_j. Moreover, $0 < a_i^N(a_j, s) < a_i^*(a_j, s)$ for all a_j.

Lemma 3. Under Assumptions 1 and 3, for $i, j = 1, 2,\ i \neq j$ and for each $s \in \mathcal{S}$, the deviation utility, $D_i^s(a_j)$, is bounded below and is a continuous, increasing, strictly concave, and differentiable function of a_j.

Statements of lemmas and technical details for Section 3

Lemma 4. Under Assumptions1-3, the set of lifetime utilities $\mathcal{V}_{s_0}$ that correspond to dynamic relational contracts is compact for each $s_0 \in \mathcal{S}$. Hence, optimal contracts exist.

It is convenient in analyzing the recursive problem to change variables and use the deviation utilities of the two agents instead of actions. Let $d_j := D_i^s(a_j)$; by Lemma 3 $\mathrm{d}D_i^s(a_j)/\mathrm{d}a_j > 0$, and we let $g_i^s(d_j) := (D_i^s)^{-1}(d_j)$. Abusing notation, surplus is $z^s(d_1, d_2) := z^s(g_2^s(d_1), g_1^s(d_2))$, with output $y^s(d_1, d_2)$ defined similarly. Given the properties of $D_i^s(a_j)$ (Lemma 3), the functions $g_j^s(d_i)$ are continuously differentiable, strictly increasing and strictly convex. Let $\mathcal{D}(s) := \{(d_1, d_2) = (D_2^s(a_1), D_1^s(a_2)) \mid (a_1, a_2) \in \mathbb{R}_+^2\}$. The contract $\{d(s^t), x(s^t)\}_{t=0}^{\infty}$ is feasible if $\sum_i x_i(s^t) \leq z^{s_t}(d(s^t))$ (total consumption does not exceed output) for every history s^t, and for actions and consumption to be non-negative, it must also satisfy $d(s^t) \in \mathcal{D}(s_t)$ for every history s^t and $x_i(s^t) + g_j^{s_t}(d_i(s^t)) \geq 0$ for $i, j = 1, 2$, $i \neq j$ and every history s^t. We define $d_i^*(s) = D_j^s(a_i^*(s))$, $i \neq j$, $d_i^*(d_j, s) = D_j^s(a_i^*(g_i^s(d_j), s))$, etc.

Problem [P1] can be reformulated with $d \in \mathcal{D}(s)$ replacing $a \geq 0$ as a choice variable, the RHS of (3.1b) and (3.1c) being d_2 and d_1 respectively, a_i in (3.1f) being $g_j^s(d_i)$ and the RHS of (3.1g) being $z^s(d_1, d_2)$, with solution denoted by $(d^s(V_1), x^s(V_1))$.[24]

With the change in variables, the first-order condition (3.2c) becomes

$$\text{(A.1)} \qquad \frac{\mu_j}{1+\mu_2} = \frac{\partial z^s}{\partial d_i}\left(u_2'(\cdot) + \frac{\gamma_2}{1+\mu_2}\right) + g_j^{s\prime}(d_i)\frac{\gamma_i}{1+\mu_2} \quad i, j = 1, 2 \text{ and } i \neq j.$$

To establish concavity of $V_2^r(\cdot)$ we give two *alternative* assumptions.

Assumption A.4. (a) $z^s(d)\colon \mathcal{D}(s) \to \mathbb{R}$ is strictly concave in d *and* (b) any solution to [P1] has $x_i > 0$ for $i = 1, 2$ and for each s^t.

Assumption A.5. The function $z^s(d) + g_j^s(d_i) : \mathcal{D}(s) \to \mathbb{R}_+$ is concave in d for each $i, j = 1, 2$, $j \neq i$.

Under either Assumption A.4 or Assumption A.5, the Pareto-frontier is concave on $[\underline{V}_1^s, \bar{V}_1^s]$ (Proposition 2(i)). Under Assumption A.4, it is easily checked that the constraint set is also convex.[25]

Although Assumptions A.4 and A.5 are not directly on primitives of the model (because they are specified in terms of the deviation utility and an endogenous variable for Assumption A.4), it is easily checked that there are natural parameterizations of the model where these assumptions are satisfied. For example, Assumption A.4 is satisfied provided that agents are not too risk averse. For example, consider the case where preferences exhibit

[24] The linear independence constraint qualification holds unless the constraints (3.1f) are inactive and $u_2'(\partial z^s/\partial a_1)(\mathrm{d}g_2^s(d_1)/\mathrm{d}d_1 0 = 1$. This constraint qualification can fail, but it only fails at $V_1 = \bar{V}_1^s$ where the slope of the Pareto-frontier is infinite (examples where the constraint qualification fails at this point can be constructed). Thus, apart from $V_1 = \bar{V}_1^s$, the linear independence constraint qualification holds and the Lagrangian multipliers in the first-order conditions (reported in sub-section 3.4) exist and are unique. We can also ignore points $V_1 = \bar{V}_1^s$ without loss of generality: if $V_1(s_0) < \bar{V}_1^{s_0}$, then we will show that $V_1 \neq \bar{V}_1^s$ for any state s; if $V_1(s_0) = \bar{V}_1^{s_0}$, then it will be possible to reformulate the problem maximizing the utility of agent 1 for a given V_2 for agent 2 and the relevant constraint qualification will be satisfied.

[25] It can also be checked that if [P1] is written with c and d as choice variables, then a sufficient condition for convexity of the constraint set is that $y^s(d)$ is concave in d. This condition is more stringent than concavity of $z^s(d)$ and will fail in a number of natural cases.

constant absolute risk aversion with coefficient $\alpha > 0$, the same for both agents, and the production function is separable and given by $y^s(a_1, a_2) = (\beta)^{-1}((a_1)^\beta + (a_2)^\beta)$ where $\beta \in (0, 1)$. Furthermore, suppose each agent can expropriate a proportion θ of output in the case of default. Then a sufficient condition for the assumption to be satisfied for $\theta \in (1/\mathrm{e}, 1/2]$ is if $\alpha < -\mathrm{e}\theta(1-\theta)^{-1}\log\theta$, and for $\theta \in (0, 1/\mathrm{e}]$ if $\alpha < (1-\theta)^{-1}$. Equally,suppose that agents are risk neutral with $u_i(x) = x$, production is additive and the breakdown consumption in each state is $\phi_i(a) = \theta_{i1} f_1(a_1) + \theta_{i2} f_2(a_2)$, where for notational simplicity the dependence of θ, f etc. on s is suppressed. With this specification for $\phi_i(a)$, $D''_j/D'_j = f''_i/f'_i$, and it can be checked that Assumption A.5 is satisfied.

Lemma 5. Under Assumptions 1-3 and under either Assumption A.4 or Assumption A.5, $d_i^s(V_1)$ is a continuous function of V_1 for each $s \in \mathcal{S}$ and $i = 1, 2$.

Lemma 6. Under Assumptions 1-3, and for $i, j = 1, 2$, $i \neq j$, for any history s^t, (i) if $V_i(s^t) > d_j(s^t)$, then $a_j(s^t) \geq a_j^*(a_i(s^t), s_t)$; (ii) if $c_i(s^t) > 0$, then $a_i(s^t) \leq a_i^*(a_j(s^t), s_t)$.

Statement of lemmas and proofs of theorem for Section 4

For all lemmas and proofs in this subsection, we maintain Assumption A.4. Additionally it is assumed that agents are risk averse, that is, u_i is strictly concave for $i = 1, 2$.

Lemma 7. For each $s \in \mathcal{S}$, a solution to [P1] has the property that $z^s(a_1, a_2)$ is maximized over $a \in \mathbb{R}_+^2$ subject to $V_1 \geq D_1^s(a_2)$ and $V_2^s(V_1) \geq D_2^s(a_1)$.

Lemma 8. For each $s \in \mathcal{S}$, the surplus function $z^s(V_1)$ is continuous, concave and differentiable in V_1.

Lemma 9. For each $s \in \mathcal{S}$, (i) $\mathrm{d}z^s(V_1)/\mathrm{d}V_1 > 0$ (< 0) implies $\mu_1^s(V_1) > 0$ $(\mu_2^s(V_1) > 0)$; (ii) there are two critical values $\bar{\chi}_1^s \in (\underline{V}_1^s, \bar{V}_1^s]$ and $\underline{\chi}_1^s \in [\underline{V}_1^s, \bar{V}_1^s)$, such that $d_2^s(V_1) = V_1$ for all $V_1 \leq \bar{\chi}_1^s$ and $d_1^s(V_1) = V_2^s(V_1)$ for all $V_1 \geq \underline{\chi}_1^s$. Moreover, $\mu_1^s(V_1) = 0$ for $\bar{V}_1^s > V_1 \geq \bar{\chi}_1^s$ and $\mu_2^s(V_1) = 0$ for $\underline{V}_1^s < V_1 \leq \underline{\chi}_1^s$ (if such V_1 exist). If the efficient actions can be sustained in state s, then $\bar{\chi}_1^s \leq \underline{\chi}_1^s$. Otherwise, $\bar{\chi}_1^s > \underline{\chi}_1^s$, and surplus is maximized for a unique value of $V_1 \in (\underline{\chi}_1^s, \bar{\chi}_1^s)$ at which both constraints bind.

Proof of Theorem 1:

Before proving the theorem, we prove two lemmas.

Since $\sigma_s^+(V_1)$ depends only on the current slope σ and the current state s (recall V_1 and σ are uniquely related for a given state) the evolution of the contract can be represented as a stochastic recursion, i.e., $\sigma(t+1) = \sigma_{s_t}^+(\sigma_{s_t}^{-1}(\sigma))$, which we write as $\sigma(t+1) = h(\sigma(t), s_t)$, and where $h\colon \mathbb{R}_+ \cup \{\infty\} \times \mathcal{S} \to \mathbb{R}_+$; $\sigma(0) = \sigma_0$ is the given initial value, corresponding to the initial state s_0 and agent 1's lifetime utility $V_1(s_0)$. (This is the same function as h defined in the text, given that $\rho(t) = \sigma(t+1)$.)

Lemma 10. (i) The function $h(\sigma, s)$ is continuous and strictly increasing in σ; (ii) for each state s, there is a single, possibly degenerate, interval of fixed points $[\underline{\sigma}_s^*, \bar{\sigma}_s^*]$, $\underline{\sigma}_s^* > 0$, such that $h(\sigma, s) = \sigma$ for any $\sigma \in [\underline{\sigma}_s^*, \bar{\sigma}_s^*]$; (iii) $h(\sigma, s) < \sigma$ for $\sigma > \bar{\sigma}_s^*$ and $h(\sigma, s) > \sigma$ for $\sigma < \underline{\sigma}_s^*$.

Proof. Let $x_i(V_1) = c_i(V_1) - g_j(d_i(V_1))$ be the net consumption of agent i (dropping the state superscript) and $\rho(V_1) := u_2'(x_2(V_1))/u_1'(x_1(V_1))$. Then, $h(\sigma, s) = \rho(\sigma_s^{-1}(\sigma))$.
We first prove part (i). From the concavity properties of [P1] under Assumption A.4, the choice variables $x_i(V_1)$ are continuous, and hence, $\rho(V_1)$ is continuous in V_1. From Proposition 2(i), the Pareto-Frontier is continuously differentiable and hence, so too is its inverse. Thus, $h(\sigma, s)$ is continuous in σ.
Next, we turn to the monotonicity of $h(\sigma, s)$. First, we show that $\rho(V_1)$ is strictly increasing. Suppose, to the contrary, that $\rho(V_1) \leq \rho(\tilde{V}_1)$ for some $V_1 > \tilde{V}_1$. It follows from $\rho(V_1) = -V_2^{r\prime}(V_1)$ and the concavity of the frontier $V_2^r(V_1)$ that $V_1^r(V_1) \leq V_1^r(\tilde{V}_1)$ for all $r \in \mathcal{S}$. Also, since $\tilde{V}_1 < V_1$, we have $u_1(x_1(\tilde{V}_1)) + \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_1^r(\tilde{V}_1) < u_1(x_1(V_1)) + \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_1^r(V_1)$. Hence, $x_1(\tilde{V}_1) < x_1(V_1)$. Likewise, since the frontier is downward sloping, $V_2(\tilde{V}_1) > V_2(V_1)$ and $V_2^r(V_1^r(\tilde{V}_1)) \leq V_2^r(V_1^r(V_1))$, and therefore, that $x_2(\tilde{V}_1) > x_2(V_1)$. But then $u_2'(x_2(\tilde{V}_1))/u_1'(x_1(\tilde{V}_1)) < u_2'(x_2(V_1))/u_1'(x_1(V_1))$ or $\rho(\tilde{V}_1) < \rho(V_1)$, which is a contradiction. Thus, we can conclude that $\rho(V_1)$ is strictly increasing in V_1. Since the frontier $V_2^s(V_1)$ is strictly decreasing in V_1 and $\sigma = -V_2^{s\prime}(V_1)$, the result is proved.
To establish parts (ii) and (iii), from Lemma 9, for $s \in \mathcal{S}_*$, surplus is at the first-best level for $V_1 \in [\bar{\chi}_1^s, \underline{\chi}_1^s]$. Correspondingly, there is an interval of Pareto frontier slopes $[\underline{\sigma}_s^*, \bar{\sigma}_s^*] := [-V_2^{s\prime}(\bar{\chi}_1^s), -V_2^{s\prime}(\underline{\chi}_1^s)]$. For $s \in \mathcal{S}_*^c$, the corresponding interval is degenerate at a single point $[\underline{\sigma}_s^*, \bar{\sigma}_s^*] := [-V_2^{s\prime}(\hat{\chi}_1^s)]$ where $\hat{\chi}_1^s = \arg\max_{V_1} z^s(V_1)$. It follows from part (b) of Assumption A.4 and equation (3.2b) (given $\gamma_i = \nu_i^r = 0$ for $i = 1, 2$) that $\sigma_s^+(V_1)$ is positive and finite. Thus, $0 < \underline{\sigma}_s^* \leq \bar{\sigma}_s^* < \infty$. Equation (4.1) therefore implies the following: If $\sigma_s(V_1) \in [\underline{\sigma}_s^*, \bar{\sigma}_s^*]$, then $\sigma_s^+(V_1) = \sigma_s(V_1)$. If $\sigma_s(V_1) > \bar{\sigma}_s^*$, then $\mathrm{d}z^s(V_1)/\mathrm{d}V_1 < 0$ (given the concavity of $z^s(V_1)$ by Lemma 8) and hence, $\sigma_s^+(V_1) < \sigma_s(V_1)$. Likewise, if $\sigma_s(V_1) < \underline{\sigma}_s^*$, then $\mathrm{d}z^s(V_1)/\mathrm{d}V_1 > 0$ and hence, $\sigma_s^+(V_1) > \sigma_s(V_1)$. ■

Let $\bar{s}$ be a state such that $\underline{\sigma}_{\bar{s}}^* \geq \underline{\sigma}_s^*$ and $\underline{s}$ a state such that $\bar{\sigma}_{\underline{s}}^* \leq \bar{\sigma}_s^*$ for all $s \in \mathcal{S}$.

Lemma 11. An FBA is sustainable if and only if $\underline{\sigma}_{\bar{s}}^* \leq \bar{\sigma}_{\underline{s}}^*$ and $\mathcal{S}_* = \mathcal{S}$.

Proof. The "if" implication follows because there would exist an initial value $\sigma_0 \in [\underline{\sigma}_{\bar{s}}^*, \bar{\sigma}_{\underline{s}}^*]$ such that $\sigma_0 \in [\underline{\sigma}_s^*, \bar{\sigma}_s^*]$ for each state s. It therefore follows that starting from σ_0, $\sigma(t)$, and hence, the ratio of marginal utilities, is kept constant at σ_0 and since surplus is maximized for $\sigma(t) \in [\underline{\sigma}_{s_t}^*, \bar{\sigma}_{s_t}^*]$, actions are CSM and thus first-best by $\mathcal{S}_* = \mathcal{S}$ in each state. "Only if" follows because by Lemma 10 even if first-best actions are sustainable in every state, $\underline{\sigma}_{\bar{s}}^* > \bar{\sigma}_{\underline{s}}^*$ would imply that whenever $s_t = \bar{s}$ and $s_\tau = \underline{s}$ (such t, τ exist with probability one given irreducibility), then either (a) $\sigma(t) \in [\underline{\sigma}_{s_t}^*, \bar{\sigma}_{s_t}^*]$ and $\sigma(\tau) \in [\underline{\sigma}_{s_\tau}^*, \bar{\sigma}_{s_\tau}^*]$ in which case $\sigma(t) > \sigma(\tau)$, and the risk-sharing condition fails, or (b) either or both $\sigma(t) \notin [\underline{\sigma}_{s_t}^*, \bar{\sigma}_{s_t}^*]$ and $\sigma(\tau) \notin [\underline{\sigma}_{s_\tau}^*, \bar{\sigma}_{s_\tau}^*]$, in which case surplus is not maximized at least one of the dates. ■

Proof of Theorem 1.
Recalling that $\rho(t) = \sigma(t+1)$, an interval $[\underline{\sigma}_s^*, \bar{\sigma}_s^*]$ corresponds to an interval of marginal utility ratios in state s and convergence of $\sigma(t)$ is equivalent to converence of $\rho(t)$. Part (a) of the Theorem therefore follows straightforwardly from Lemmas 10 and 11. From Lemma 11 $\underline{\sigma}_{\bar{s}}^* \leq \bar{\sigma}_{\underline{s}}^*$. Convergence is to $\underline{\sigma}_{\bar{s}}^*$ if $\sigma_0 < \underline{\sigma}_{\bar{s}}^*$, since $\sigma(t) = h(\sigma(t-1), s_{t-1}) \geq \sigma(t-1)$ by Lemma 10 (iii) and so $\{\sigma(t)\}$ is a non-decreasing sequence; it is bounded above by $\underline{\sigma}_{\bar{s}}^*$ given h continuous and increasing in σ and that $h(\sigma, s) \leq \sigma$, all s, for $\sigma \geq \underline{\sigma}_{\bar{s}}^*$, by Lemma 10 (ii)

and (iii); with probability one $\sigma(t)$ converges to $\underline{\sigma}^*_{\bar{s}}$ given that $h(\sigma, \bar{s}) > \sigma$ for $\sigma < \underline{\sigma}^*_{\bar{s}}$ and the irreducibility of $[\pi_{sr}]$ and finiteness of states implies that state $\bar{s}$ is recurrent. Likewise convergence (monotonic) is to $\bar{\sigma}^*_{\underline{s}}$ if $\sigma_0 > \bar{\sigma}^*_{\underline{s}}$, and $\sigma(t)$ is constant at σ_0 if $\sigma_0 \in [\underline{\sigma}^*_{\bar{s}}, \bar{\sigma}^*_{\underline{s}}]$. If there exist multiple FBAs then $\underline{\sigma}^*_{\bar{s}} < \bar{\sigma}^*_{\underline{s}}$, and the limit depends on σ_0 and hence on $V_1(s_0)$. Since $\lim_{t\to\infty} \sigma(t) \in [\underline{\sigma}^*_s, \bar{\sigma}^*_s]$ for all s, $\rho(t)$ converges and by continuity the limit actions are $a^*(s_t)$, and $x^*(s_t)$ is such that $u_2'(x_2^*(s))/u_1'(x_1^*(s)) = \lim_{t\to\infty} \sigma(t)$, all s.
For (b), if an FBA is not sustainable, then by Lemma 11 either $\underline{\sigma}^*_{\bar{s}} = \bar{\sigma}^*_{\underline{s}}$ and the CSM actions are below first-best levels in at least one state, or $\underline{\sigma}^*_{\bar{s}} > \bar{\sigma}^*_{\underline{s}}$. In the former case by Lemma 10, following the argument in part (a), $\{\sigma(t)\}$ is monotonic and converges with probability one to $\underline{\sigma}^*_{\bar{s}} = \bar{\sigma}^*_{\underline{s}} \in [\underline{\sigma}^*_s, \bar{\sigma}^*_s]$ for all s, implying that limiting actions are CSM in each state, establishing case (b)(i). Otherwise, part (b)(ii) obtains; $\sigma(t) \in [\min_s\{h(0,s)\}, \max_s\{h(\infty,s)\}]$ for $t \geq 1$. Irreducibility and finiteness of $[\pi_{sr}]$ implies $\underline{s}$ and $\bar{s}$ are recurrent, and $\underline{\sigma}^*_{\bar{s}} > \bar{\sigma}^*_{\underline{s}}$ implies that for any $\sigma \geq 0$, either $h(\sigma, \bar{s}) > \sigma$ or $h(\sigma, \underline{s}) < \sigma$ (or both). Thus, given h is continuous in σ, weak convergence to a degenerate distribution is impossible. Next consider the sequence of r.v.s $\{a(t) - \bar{a}(s_t)\}$. Assume w.l.o.g. that state $\underline{s}$ is uniquely defined. Consider an infinite history $\{s_0, s_1, \ldots\}$ in which each state occurs infinitely often, which implies from the properties of h established in Lemma 10 that there exists t' such that $\sigma(t) \geq \bar{\sigma}^*_{\underline{s}}$ for $t \geq t'$; note that the set of such histories has probability one. Suppose that $a(t) - \bar{a}(s_t) \to 0$, so that along the subsequence $\{s_{t_1}, s_{t_2}, s_{t_3}, \ldots\}$ where t_i is the ith time $\underline{s}$ occurs, $a(t_i) \to \bar{a}(\underline{s})$ as $i \to \infty$. Consider a $t \geq t'$ such that $s_t = \bar{s}$. Then $\sigma(t) \geq h(\bar{\sigma}^*_{\underline{s}}, \bar{s}) > \bar{\sigma}^*_{\underline{s}}$ by h increasing in σ and $h(\sigma, \bar{s}) > \sigma$ for $\sigma < \underline{\sigma}^*_{\bar{s}}$. If t_i is the next time $\underline{s}$ occurs, $\sigma(t_i) \geq \min\{h(\bar{\sigma}^*_{\underline{s}}, \bar{s}), \min_{s \neq \underline{s}} \bar{\sigma}^*_s\} > \bar{\sigma}^*_{\underline{s}}$. This implies that $V_1(t_i)$ is bounded above $\arg\max_{V_1} z^{\underline{s}}(V_1)$, i.e., above $\sigma_{\underline{s}}^{-1}(\bar{\sigma}^*_{\underline{s}})$, so $a(t_i)$ is bounded away from $\bar{a}(\underline{s})$. Since $s_t = \bar{s}$ infinitely often, this contradicts $a(t_i) \to \bar{a}(\underline{s})$. Next, fix any $\sigma_c \in (\bar{\sigma}^*_{\underline{s}}, \underline{\sigma}^*_{\bar{s}})$; clearly $h(\sigma_c, \bar{s}) > \sigma_c$ and $h(\sigma_c, \underline{s}) < \sigma_c$. Using $\pi_{ss} > 0$ all s, there exist $t \geq 1$ such that

$$\varepsilon_1 := \mathbb{P}(\sigma(t) < \sigma_c | \; \sigma_0 = (\max_s\{h(\infty, s)\})) > 0$$
$$\varepsilon_2 := \mathbb{P}(\sigma(t) > \sigma_c | \; \sigma_0 = (\min_s\{h(0, s)\}) > 0,$$

since for ε_1 (respectively ε_2) consider a sufficient number of consecutive occurrences of $\underline{s}$ (respectively $\bar{s}$). This implies the "splitting condition" of Bhattacharya and Majumdar (2007; Chapter 3.5, p250) for the i.i.d. case, and the condition in Foss et al. (2018; Corollary 1) in the general Markov case. Thus, there is a unique stationary distribution $\tilde{F}$ such that $F_t^{(V(s_0))}$ converges weakly to $\tilde{F}$, as $t \to \infty$, for any initial condition. ■

Statement of lemmas and proof of theorem for Section 5

For this subsection we maintain Assumption A.5 but additionally assume that agents are risk-neutral with $u_i(x_i) = x_i$.

Lemma 12. For each $s \in \mathcal{S}$, the Pareto-frontier $V_2^s(\cdot)$ is strictly concave on $[\underline{V}_1^s, \underline{V}_1^{s*})$ where $\underline{V}_1^{s*} := \inf\{V_1 : V_2^{s\prime}(V_1) = -1\}$, and on $(\bar{V}_1^{s*}, \bar{V}_1^s]$ where $\bar{V}_1^{s*} := \sup\{V_1 : V_2^{s\prime}(V_1) = -1\}$. If first-best actions are not sustainable in state s, i.e., for $s \in \mathcal{S}^c_*$, then $V_2^s(\cdot)$ is strictly concave on $[\underline{V}_1^s, \bar{V}_1^s]$.

Lemma 13. With probability one, there is a random time $\hat{t} < \infty$ such that $\zeta(t)$ converges monotonically to 0 with $\zeta(t) = 0$ for all $t \geq \hat{t} - 1$.

Lemma 14. For each $s \in \mathcal{S}$, the surplus function $z^s(V_1)$ is a continuous and single peaked function of V_1. That is, for any $V_1^{(1)} < V_1^{(2)} < V_1^{(3)}$, it is not possible that $z^s(V_1^{(1)}), z^s(V_1^{(3)}) > z^s(V_1^{(2)})$. Moreover, $z^s(V_1)$ is maximal when $\sigma_s(V_1) = 1$.

Proof of Theorem 2.
Stationary phase: Define $\hat{t}$ as the earliest date at which $\sigma(t) = 1$. By Lemma 13, $\hat{t} < \infty$ with probability 1. If $\sigma = 1$ and $V_1 \in A$ (so that $c_1 = 0$), then it follows from the first-order conditions that $\gamma_1 = \gamma_2 = \mu_1 = \mu_2 = 0$, and hence that $a(t) = a^*(s_t)$ or equivalently $d(t) = d^*(s_t)$ (we suppress t below for notational simplicity). A similar argument applies for $\sigma = 1$ and $V_1 \in C$. If $\sigma = 1$ and $V_1 \in B$ (so $c_1 > 0$ and $c_2 > 0$), then $\mu_1 = \mu_2$ from (3.2a). Thus, either $\mu_i = 0$, $i = 1, 2$, in which case again $d = d^*$ and $z^s(d)$ is maximal, or $\mu_i > 0$, $i = 1, 2$, so both self-enforcing constraints bind. In the latter case, $(\partial z^s/\partial d_2)/(\partial z^s/\partial d_1) = 1 = -V_2^{s\prime}(V_1)$, from (3.2a) and (A.1). From the concavity of $V_2^s(V_1)$ and $z^s(d)$, this implies that $z^s(d)$ is maximized by choice of $d \in \mathcal{D}(s)$ and $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$ subject to $d_2 \leq V_1, d_1 \leq V_2(V_1)$. Since these self-enforcing constraints must hold for any dynamic relational contract , it follows that at $\sigma = 1$, $z^s(d)$ is maximal across all dynamic relational contracts whether the self-enforcing constraints bind or not and optimal actions are CSM. Also in the case where $\mu_1 = \mu_2 > 0$, it follows from (A.1) that $d_i < d_i^*(d_j, s) \leq d_i^*(s)$, $j \neq i$ (the last inequality follows because $d_i^*(d_j, s)$ is non-decreasing in d_j). Since a_i and d_i are positively monotonically related through the function g_j^s and $a = a^*$ if and only if $d = d^*$, the statement in the Theorem follows.
Backloading phase: Suppose $V_1(s_0)$ is such that $\sigma_0 < 1$ (a symmetric argument applies if $\sigma_0 > 1$). Then $\sigma(t) \leq \sigma^+(t) \equiv \sigma(t+1) \leq 1$ and $\gamma_2(t) = 0$.
We first establish the last part of the theorem. Consider $t = \hat{t} - 1$, so that $\sigma^+(t) \equiv \sigma(t+1) = 1$. Then $\sigma^+(t) - \sigma(t) > 0$ and from (3.2a), $\mu_1(t) > 0$. (3.2b) implies that $\gamma_1(t) = 0$. So from (A.1), $\partial z^{s_t}/\partial d_1 \geq 0$, and thus, $d_1(t) \leq d_1^*(d_2(t), s_t)$. Likewise in (A.1), $d_2(t) < d_2^*(d_1(t), s_t)$. Together with $d_1(t) \leq d_1^*(d_2(t), s_t)$ this implies $d_1(t) \leq d_1^*(s_t)$ and $d_2(t) < d_2^*(s_t)$; equivalently $a_1(t) \leq a_1^*(s_t)$ and $a_2(t) < a_2^*(s_t)$.
Next, suppose $\sigma^+(t) < 1$, so that $t < \hat{t}-1$ and from (5.1a) $\gamma_1 > 0$ (so $c_1 = 0$ and $V_1(t) \in A^{s_t}$). Equation (5.1a) implies that $\partial y^s(a_1, a_2)/\partial a_1 < 1$, so that $a_1(t) > a_1^*(a_2(t), s_t)$. Again using (A.1), $\partial z^s(d_1, d_2)/\partial d_2 \geq 0$, so that $d_2(t) \leq d_2^*(d_1(t), s_t)$ and hence $a_2(t) \leq a_2^*(a_1(t), s_t)$.
To establish the monotonicity of the marginal conditions, consider dates t and t' with $\hat{t} \geq t' > t$ such that the same state s occurs at date t and t'. If $\sigma^+(t') < 1$, then it follows from the monotonicity of the sequence established in Lemma 13 that $\sigma^+(t) \leq \sigma^+(t') < 1$. Hence, $V_1(t) \in A^{s_t}$ and $V_1(t') \in A^{s_{t'}}$. It follows directly from (5.1a) that $1 > \partial y^s(a(t'))/\partial a_1 \geq \partial y^s(a(t))/\partial a_1$. Now suppose, contrary to the assertion that $\partial y^s(a(t'))/\partial a_2 > \partial y^s(a(t))/\partial a_2$ or equivalently $\partial z^s(a(t'))/\partial a_2 > \partial z^s(a(t))/\partial a_2$. From (5.1a) and (5.1b) $\partial z^s(a(t))/\partial a_2 (\mathrm{d}D_1^s/\mathrm{d}a_2)^{-1} \equiv \partial z^s(d(t))/\partial d_2 = \sigma^+(t) - \sigma(t) \geq 0$, and hence $\partial z^s(a(t))/\partial a_2 \geq 0$ and $\partial z^s(a(t'))/\partial a_2 > 0$. Strict concavity of $z^s(a)$ requires that $\sum_{i=1}^{2}((\partial z^s(a(t')/\partial a_i) - (\partial z^s(a(t)/\partial a_i))(a_i(t') - a_i(t)) < 0$ for $a(t) \neq a(t')$. Then, since $\partial^2 z^s/\partial a_1 \partial a_2 \geq 0$, it follows that $a_1(t) \geq a_1(t')$ and $a_2(t) > a_2(t')$. This however provides a contradiction. To see this, consider that $\sigma(t') \geq \sigma(t)$ and

$\sigma(t) < 1$ imply, from Lemma 12, that $V_1(t') \geq V_1(t)$. Equally, because $\partial z^s(a(t'))/\partial a_2 > 0$, $\mu_1(t') > 0$, and hence, $d_2(t') = V_1(t') \geq V_1(t) \geq d_2(t)$. This implies $a_2(t') \geq a_2(t)$, a contradiction. A similar argument applies if $\sigma^+(t) < \sigma^+(t') = 1$ except that in this case we have $\gamma_1(t') = \gamma_2(t') = 0$ and thus, from (A.1), $\partial z^s(a(t'))/\partial a_1 \geq 0 > \partial z^s(a(t))/\partial a_1$ (the second inequality follows from the earlier argument because $\sigma^+(t) < 1$). Finally, Lemma 14 shows that $z^s(V_1)$ is continuous and single-peaked and has a maximum when $\sigma_s(V_1) = 1$. From above, $V_1(t') \geq V_1(t)$ when $\sigma(t) < 1$. Hence, we conclude that $z^s(a(t)) \leq z^s(a(t'))$. ■

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Dynamic Relational Contracts Under Complete Information Supplementary Material

This Supplementary Material contains the statements and omitted proofs of all propositions from the paper "Dynamic Relational Contracts under Complete Information", by Jonathan P. Thomas and Tim Worrall, *Journal of Economic Theory*, 2018, doi:10.1016/j.jet.2018.02.004. It also includes the statements and proofs of those lemmas given without proof in the Appendix of the paper.

Proofs of lemmas for Section 2

Lemma 1. Under Assumption 2, for $i, j = 1, 2$, $i \neq j$ and for each $s \in \mathcal{S}$, the conditionally efficient action, $a_i^*(a_j, s)$, is single-valued, weakly increasing and continuous in a_j.

Proof. By Assumption 2, holding a_j fixed, $y^s(a_1, a_2)$ is strictly concave in a_i. Thus, the conditionally efficient actions are uniquely defined. From the continuity and differentiability assumptions, each $a_i^*(a_j, s)$ is a continuous function of a_j. Complementarity in production implies that $a_i^*(a_j, s)$ is weakly increasing in a_j for $i, j = 1, 2$, $i \neq j$. ■

Lemma 2. Under Assumption 3, for $i, j = 1, 2$, $i \neq j$ and for each $s \in \mathcal{S}$, the Nash best-response, $a_i^N(a_j, s)$, is single-valued, weakly increasing and continuous in a_j. Moreover, $0 < a_i^N(a_j, s) < a_i^*(a_j, s)$ for all a_j.

Proof. Since the lemma applies for any given state s, the notational dependence on s can be dropped. Uniqueness of the $a_i^N(a_j)$ follows from Assumption 3 that $\phi_i(a_i, a_j)$ is strictly concave in its own action. Standard results imply these are continuous and differentiable functions. Since, from Assumption 3, $\partial\phi_i(0, a_j)/\partial a_i > 1$, it follows that $a_i^N(a_j) > 0$. Moreover, from the inequality in (2.1), $1 < \partial\phi_i(0, a_j)/\partial a_i < \partial y(0, a_j)/\partial a_i$, so that $a_i^*(a_j) > 0$. Thus,

$$1 = \frac{\partial\phi_i(a_i^N(a_j), a_j)}{\partial a_i} = \frac{\partial y(a_i^*(a_j), a_j)}{\partial a_i} > \frac{\partial\phi_i(a_i^*(a_j), a_j)}{\partial a_i},$$

where the first two equalities hold from the first-order conditions for $a_i^N(a_j)$ and $a_i^*(a_j)$ respectively, and the last inequality follows from (2.1). It then follows from the strict concavity of ϕ_i in its own argument (Assumption 3), that $a_i^N(a_j) < a_i^*(a_j)$. ■

Lemma 3. Under Assumptions 1 and 3, for $i, j = 1, 2$, $i \neq j$ and for each $s \in \mathcal{S}$, the deviation utility, $D_i^s(a_j)$, is bounded below and is a continuous, increasing, strictly concave, and differentiable function of a_j.

Proof. Using Lemma 2 and the definition of the deviation utility establishes its continuity and differentiability. The derivative satisfies:

$$D_1^{s\prime}(a_2) = u_1'(\phi_1^s(a_1^N(a_2, s), a_2) - a_1^N(a_2, s))\frac{\partial\phi_1^s(a_1^N(a_2, s), a_2)}{\partial a_2}.$$

Thus, $D_1^s(a_2)$ is strictly increasing in a_2 by the hold-up assumption in Assumption 3. To show it is strictly concave, let $\upsilon_1^N(a_2, s) := \max_{\tilde{a}_1} \phi_1^s(\tilde{a}_1, a_2) - \tilde{a}_1$. Dropping the state notation, since nothing depends on it, consider two values $a_2 \neq \hat{a}_2$ and the convex combination $a_2^\lambda = \lambda a_2 + (1-\lambda)\hat{a}_2$ for $\lambda \in (0, 1)$. Then,

$$\begin{aligned}
\upsilon_1^N(a_2^\lambda) &= \phi_1(a_1^N(a_2^\lambda), a_2^\lambda) - a_1^N(a_2^\lambda) \\
&\geq \phi_1(\lambda a_1^N(a_2) + (1-\lambda)a_1^N(\hat{a}_2), a_2^\lambda) - \left(\lambda a_1^N(a_2) + (1-\lambda)a_1^N(\hat{a}_2)\right) \\
&> \lambda\left(\phi_1(a_1^N(a_2), a_2) - a_1^N(a_2)\right) + (1-\lambda)\left(\phi_1(a_1^N(\hat{a}_2), \hat{a}_2) - a_1^N(\hat{a}_2)\right) \\
&= \lambda\upsilon_1^N(a_2) + (1-\lambda)\upsilon_1^N(\hat{a}_2),
\end{aligned}$$

where the first inequality follows from optimality and the second strict inequality from Assumption 3 that $\phi_1(a)$ is strictly concave. Since $D_1(a_2) = u_1(v_1^N(a_2)) + \delta \sum_{r \in \mathcal{S}} \pi_{sr} D_1^r(a_2^{NE}(r))$ and u_1 is itself concave, it follows that $D_1(a_2)$ is strictly concave. From Assumption 3, $v_1^N(a_2) \geq v_1^N(0) > 0$. Therefore, from Assumption 1, $u_1(v_1^N(a_2)) > -\infty$. Likewise, $D_1^r(a_2^{NE}(r)) > -\infty$, and hence, $D_1(a_2)$ is bounded below too. The same arguments apply with agent indices reversed. ■

Proofs of lemmas and propositions for Section 3

Lemma 4. Under Assumptions1-3, the set of lifetime utilities $\mathcal{V}_{s_0}$ that correspond to dynamic relational contracts is compact for each $s_0 \in \mathcal{S}$. Hence, optimal contracts exist.

Proof. By Lemma 3, $D_i^s(a_j)$ is bounded below and by Assumption 2, $z^s(a)$ is bounded above for each $s \in \mathcal{S}$. Together these facts imply that the future utility for agent i is bounded above. Therefore, in order to satisfy (2.2), it follows that $u_i(x_i(s^t))$, and hence $x_i(s^t)$, is above some bound, say $\hat{x}_i$, at each s^t. By Assumption 2, the set of actions $\mathcal{A}(s) = \{(a_1, a_2) \in \mathbb{R}_+^2 \mid z^s(a) \geq \hat{x}_1 + \hat{x}_2\}$ is compact. Therefore, the action-consumption pairs after any history s^t can be restricted to a compact subset, say $\digamma(s^t) \subset \mathbb{R}^4$. Hence, the product space $\prod_{s^t} \digamma(s^t)$ is sequentially compact in the product topology because it is a countable product of compact spaces. Associated with any dynamic relational contract (and for notational simplicity, ignoring the dependence on the initial state) is a pair of discounted utilities (V_1, V_2). Let Γ denote the set of dynamic relational contracts and $\mathcal{V}$ the set of associated discounted utilities. Consider any convergent sequence in $\mathcal{V}$ and the associated sequence of dynamic relational contracts in Γ. By sequential compactness, the latter has a convergent sub-sequence that converges pointwise to some limiting contract. By the dominated convergence theorem, the limit of the sequence of utilities at each s^t along the subsequence must satisfy the self-enforcing constraints (2.2) because utilities are continuous functions of contracts in this topology when $\delta < 1$, and because the constraints are weak inequalities. Thus, the limit contract is a dynamic relational contract , and the limiting sequence of the associated lifetime utilities has a limit point that corresponds to the limit dynamic relational contract . It follows that $\mathcal{V}$ is closed and bounded, and hence, a compact subset of $\mathbb{R}^2$. The existence of optimal contracts then follows by maximizing weighted sums (with non-negative weights) of utilities over this set. ■

Proposition 1. In any optimal contract (i) actions are never below the Nash reaction functions, $a_i(s^t) \geq a_i^N(a_j(s^t), s_t)$, and $a(s^t) \geq a^{NE}(s_t) > 0$; (ii) an agent who is allocated all current output and who is not overinvesting, $a_i(s^t) \leq a_i^*(a_j(s^t), s_t)$, is unconstrained.

Proof. We drop the state notation because nothing depends on it. (i) We first note that $a^{NE} > 0$ because, from Lemma 2, $a_i^N(a_j) > 0$ for all a_j. The proof proceeds in two parts. The first is to show that one cannot simultaneously have $a_2 < a_2^N(a_1)$ and $a_1 \geq a_1^N(a_2)$ or vice-versa. Thus, the actions must be either above both reaction functions or below both reaction functions. The next part shows that $(a_1, a_2) \geq (a_1^{NE}, a_2^{NE})$. Since the reaction functions are non-decreasing from Lemma 2, this rules out that both actions are below the reaction functions. Step 1: Suppose that at some date t, $a_2 < a_2^N(a_1)$ and $a_1 \geq a_1^N(a_2)$. Then

$$\frac{\partial \phi_2(a_1, a_2)}{\partial a_2} > \frac{\partial \phi_2(a_1, a_2^N(a_1))}{\partial a_2} = 1 \tag{S.1}$$

since ϕ_2 is strictly concave, and

$$\frac{\partial \phi_1(a_1, a_2)}{\partial a_2} \geq \frac{\partial \phi_1(a_1^N(a_2), a_2)}{\partial a_2}, \tag{S.2}$$

by complementarity, given $a_1 \geq a_1^N(a_2)$. Consider a small increase in a_2 of $\Delta a_2 > 0$. The consequent increase in output is approximately $(\partial y(a_1, a_2)/\partial a_2)\Delta a_2$. If the self-enforcing constraint of agent 1 is not binding, this increase in output can be given to agent 2 without violating any constraints. Suppose then, that agent 1's self-enforcing constraint is binding. Change the contract by increasing agent 1's consumption at date t, so that her utility increases by the same amount as the increase in her deviation utility. From the envelope theorem, the increase in the deviation utility is, to a first-order approximation, $D_1'(a_2)\Delta a_2 = u_1'(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2))(\partial\phi_1(a_1^N(a_2),a_2)/\partial a_2)\Delta a_2$. The remainder of the extra output (we show this is positive below because agent 2 will be better off) is given to agent 2. Keep the future unchanged. We now show that these changes meet the constraints and lead to a Pareto-improvement, contrary to the assumed optimality of the contract. Let w_i denote the current utility of agent i. First, agent 1 is no worse off (in fact better off, given the hold-up assumption) and by construction her self-enforcing constraint is satisfied. For agent 2, the change in current utility is, to a first-order approximation,

$$\Delta w_2 \simeq u_2'(c_2 - a_2)\left(\frac{\partial y(a_1,a_2)}{\partial a_2} - \frac{u_1'(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2))}{u_1'(c_1 - a_1)}\frac{\partial\phi_1(a_1^N(a_2),a_2)}{\partial a_2} - 1\right)\Delta a_2. \tag{S.3}$$

Since agent 1's self-enforcing constraint is binding, $V_1 = D_1(a_2)$ and therefore $u_1(c_1 - a_1) + \delta\sum_{r\in\mathcal{S}}\pi_{sr}V_1^r = u_1(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2)) + \delta\sum_{r\in\mathcal{S}}\pi_{sr}D_1^r(a_2^{NE}(r))$. Also, since $V_1^r \geq D_1^r(a_2^{NE}(r))$, $u_1(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2)) \geq u_1(c_1 - a_1)$. Therefore, it follows that $u_1'(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2)) \leq u_1'(c_1 - a_1)$. Using this, the fact that $\partial y(a_1,a_2)/\partial a_2 \geq \sum_i^2 \partial\phi_i(a_1,a_2)/\partial a_2$, from Assumption 3, the inequality in (S.2) above, gives

$$\frac{\partial y(a_1,a_2)}{\partial a_2} - \frac{u_1'(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2))}{u_1'(c_1 - a_1)}\frac{\partial\phi_1(a_1^N(a_2),a_2)}{\partial a_2} \geq \frac{\partial y(a_1,a_2)}{\partial a_2} - \frac{\partial\phi_1(a_1,a_2)}{\partial a_2} \geq \frac{\partial\phi_2(a_1,a_2)}{\partial a_2}.$$

Then using (S.1), the bracketed term in (S.3) satisfies:

$$\left(\frac{\partial y(a_1,a_2)}{\partial a_2} - \frac{u_1'(\phi_1(a_1^N(a_2),a_2) - a_1^N(a_2))}{u_1'(c_1 - a_1)}\frac{\partial\phi_1(a_1^N(a_2),a_2)}{\partial a_2} - 1\right) > 0.$$

Thus, for Δa_2 small enough, $\Delta w_2 > 0$. Agent 2's constraint is satisfied because a_1, and hence also $D_2(a_1)$ are unchanged, while his utility has risen, so a Pareto-improvement has been demonstrated. A symmetric argument applies when $a_1 < a_1^N(a_2)$ and $a_2 \geq a_2^N(a_1)$.
Step 2: Suppose that $(a_1,a_2) \leq (a_1^{NE},a_2^{NE})$ with strict inequality for at least one agent, say agent 2. Consider replacing the actions with the Nash equilibrium actions a_i^{NE}, so that output rises from $y(a_1,a_2)$ to $y(a_1^{NE},a_2^{NE})$. Let agent 1 have consumption of $\phi_1(a_1^{NE},a_2^{NE})$ and give the remainder of the output to agent 2 (we shall show that utility does not fall, so consumption does not fall, and thus, the change is feasible). The change in per-period utilities are

(S.4)

$$\Delta w_1 = u_1(\phi_1(a_1^{NE},a_2^{NE}) - a_1^{NE}) - u_1(c_1 - a_1)$$
$$\Delta w_2 = u_2(y(a_1^{NE},a_2^{NE}) - \phi_1(a_1^{NE},a_2^{NE}) - a_2^{NE}) - u_2(c_2 - a_2) \geq u_2(\phi_2(a_1^{NE},a_2^{NE}) - a_2^{NE}) - u_2(c_2 - a_2).$$

By the definition of (a_1^{NE},a_2^{NE}), $D_i(a_j^{NE}) = u_i(\phi_i(a_1^{NE},a_2^{NE}) - a_i^{NE}) + \delta\sum_{r\in\mathcal{S}}\pi_{sr}D_i(a_{j,r}^{NE},r)$ for $i = 1,2$, $i \neq j$. Hence, for agent 1

$$D_1(a_2^{NE}) - D_1(a_2) = u_1(\phi_1(a_1^{NE},a_2^{NE}) - a_1^{NE}) - u_1(\phi_1(a_1^N(a_2),a_2)) - a_1^N(a_2))), \tag{S.5}$$

with a similar expression for agent 2. Using the expression for Δw_1 in (S.4) and (S.5) gives,

$$\begin{aligned}\Delta w_1 - (D_1(a_2^{NE}) - D_1(a_2)) &= \big(u_1(\phi_1(a_1^{NE},a_2^{NE}) - a_1^{NE}) - u_1(c_1 - a_1)\big) - \\ &\quad \big(u_1(\phi_1(a_1^{NE},a_2^{NE}) - a_1^{NE}) - u_1(\phi_1(a_1^N(a_2),a_2)) - a_1^N(a_2)))\big) \\ &= u_1(\phi_1(a_1^N(a_2),a_2)) - a_1^N(a_2)) - u_1(c_1 - a_1).\end{aligned} \tag{S.6}$$

We can assume that $V_1 = D_1(a_2)$, otherwise it would be possible to raise a_2 and reallocate output in a Pareto-improving way. Thus, by the same arguments as in Step 1, $u_1(\phi_1(a_1^N(a_2), a_2) - a_1^N(a_2)) \geq u_1(c_1 - a_1)$, so that $\Delta w_1 - (D_1(a_2^{NE}) - D_1(a_2)) \geq 0$: the change does not violate the self-enforcing constraint of agent 1. Moreover, since $a_2^{NE} > a_2$ by assumption, and since Lemma 3 shows that $D_1(a_2)$ is strictly increasing, it follows from (S.6) that $\Delta w_1 > 0$. Now consider agent 2. The new consumption of agent 2 is equal to $y(a_1^{NE}, a_2^{NE}) - \phi_1(a_1^{NE}, a_2^{NE})$, which by Assumption 3 is at least $\phi_2(a_1^{NE}, a_2^{NE})$. Thus, the change in current utility of agent 2 satisfies $\Delta w_2 = u_2(\phi_2(a_1^{NE}, a_2^{NE}) - a_1^{NE}) - u_2(c_2 - a_2)$ and the same argument as above can be applied to show $\Delta w_2 - (D_2(a_1^{NE}) - D_2(a_1)) \geq 0$. Thus, we obtain a contradiction to the assumed optimality of the original contract.

(ii) Suppose $c_1 = y(a_1, a_2)$. From part (i), optimal actions satisfy $a_1 \geq a_1^N(a_2)$. Thus, $a_1^N(a_2) \leq a_1 \leq a_1^*(a_2)$ and $y(a_1, a_2) - a_1 \geq y(a_1^N(a_2), a_2) - a_1^N(a_2)$. Equally, by Assumption 3, $a_1^N(a_2) > 0$, so that $y(a_1^N(a_2), a_2) - a_1^N(a_2) > \phi_1(a_1^N(a_2), a_2) - a_1^N(a_2)$. Thus, $u_1(y(a_1, a_2) - a_1) > u_1^N(a_2)$. Write V_1^r for next-period continuation utility in state r. Then,

$$\begin{aligned} D_1(a_2) &:= u_1^N(a_2) + \delta \sum_r \pi_{sr} D_1^r(a_2^{NE}(r)) \\ &< u_1(y(a_1, a_2) - a_1) + \delta \sum_r \pi_{sr} V_1^r, \end{aligned}$$

where the inequality follows from $u_1(y(a_1, a_2) - a_1) > u_1^N(a_2)$ and $V_1^r \geq D_1^r(a_2^{NE}(r))$. ■

Proposition 2. For each $s \in \mathcal{S}$ and under Assumptions 1-3 (i) under either Assumption A.4 or Assumption A.5, $V_2^s(V_1)$ is a continuous and concave function of V_1 defined on a non-degenerate closed interval $[\underline{V}_1^s, \bar{V}_1^s]$,and is continuously differentiable on its interior. Moreover,

$$V_2^{s(+)}(\underline{V}_1) = 0 \quad \text{and} \quad V_2^{s(-)}(\bar{V}_1) = -\infty,$$

where $V_2^{s(+)}$ denotes the right and $V_2^{s(-)}$ the left derivative. (ii) Under Assumption A.4, $V_2^s(V_1)$ is *strictly* concave if u_i is strictly concave, $i = 1, 2$, or over any interval such that $a^s(V_1)$ varies with V_1; under Assumption A.5, $V_2^s(V_1)$ is strictly concave over any interval such that $a^s(V_1)$ varies with V_1.

Proof. We first note that Assumption A.5 implies that $z^s(d)$ is strictly concave. Since $z^s(d) + g_j^s(d_i)$ is concave, $z^s(d^\lambda) - (\lambda z^s(d) + (1-\lambda) z^s(\hat{d}) \geq (\lambda g_j^s(d_i) + (1-\lambda) g_j^s(\hat{d}_i)) - g_j^s(d_i^\lambda)$ for pairs d and $\hat{d}$ and $d^\lambda = \lambda d + (1-\lambda)\hat{d}$ and $\lambda \in [0, 1]$. Since $g_j^s(d_i)$ is strictly convex, $(\lambda g_j^s(d_i) + (1-\lambda) g_j^s(\hat{d}_i)) - g_j^s(d_i^\lambda) > 0$ for $\lambda \in (0, 1)$ and $d_i \neq \hat{d}_i$ and $i, j = 1, 2$, $i \neq j$. Hence, for $d \neq \hat{d}$ and $\lambda \in (0, 1)$, we have $z^s(d^\lambda) - (\lambda z^s(d) + (1-\lambda) z^s(\hat{d}) > 0$, so that $z^s(d)$ is strictly concave.

First consider Assumption A.5. Let $(x(s^t), d(s^t))_{t=0}^\infty$ and $(\widehat{x}(s^t), \widehat{d}(s^t))_{t=0}^\infty$ be two optimal contracts with utilities of $(V_1, V_2^s(V_1))$ and $(\widehat{V}_1, V_2^s(\widehat{V}_1))$ respectively, with $V_1 \neq \widehat{V}_1$ (if there is a unique optimal contract, i.e., a unique Pareto-efficient allocation, then the lemma is trivial). Take a convex combination of the two contract actions, such that $d^\lambda(s^t) = \lambda d(s^t) + (1-\lambda)\widehat{d}(s^t)$ is chosen each period, $0 < \lambda < 1$. Define, for $i, j = 1, 2$, $i \neq j$,

$$h_j^{s_t}(s^t) := g_j^{s_t}(d_i^\lambda(s^t)) - \left(\lambda g_j^{s_t}(d_i(s^t)) + (1-\lambda) g_j^{s_t}(\widehat{d}_i(s^t)\right).$$

Since g_j^s is convex, $h_j^s \leq 0$. We want to show that it is feasible to choose $\tilde{x}_i(s^t)$ such that $\tilde{c}_i(s^t) \equiv \tilde{x}_i(s^t) + g_j^{s_t}(d_i^\lambda(s^t)) \geq 0$, $\tilde{x}_1(s^t) + \tilde{x}_2(s^t) \leq z^s(d^\lambda(s^t))$, equivalently $\tilde{c}_1(s^t) + \tilde{c}_2(s^t) \leq y^s(d^\lambda(s^t))$, and $\tilde{x}_i(s^t) \geq \lambda x_i(s^t) + (1-\lambda)\widehat{x}_i(s^t)$, equivalently $\tilde{c}_i(s^t) \geq (\lambda c_i(s^t) + (1-\lambda)\widehat{c}_i(s^t)) + h_j(s^t)$, $i = 1, 2$ (where c_i and $\widehat{c}_i$ are the consumptions corresponding to the original contracts). For notational convenience, we drop the dependence on the state and history for the moment. Let $\tilde{c}_1 = \max\{\lambda c_1 + (1-\lambda)\widehat{c}_1 + h_2, 0\}$ and $\tilde{c}_2 = y(d^\lambda) - \tilde{c}_1$. There are two cases to consider: case I, where $\lambda c_1 + (1-\lambda)\widehat{c}_1 + h_2 < 0$, and case II, where $\lambda c_1 + (1-\lambda)\widehat{c}_1 + h_2 \geq 0$.
Case I. In this case, $\tilde{c}_1 = 0$ and $\tilde{c}_2 = y(d^\lambda)$. Then, by assumption, $\tilde{c}_1 > \lambda c_1 + (1-\lambda)\widehat{c}_1 + h_2$. Furthermore, $\tilde{c}_2 > 0$ since actions, and hence, output is positive. Next $\tilde{c}_2 - (\lambda c_2 + (1-\lambda)\widehat{c}_2 + h_1) \geq y(d^\lambda) - (\lambda y(d) +$

$(1-\lambda)y(\widehat{d})+h_1) \geq 0$, where the first inequality follows because $c_2 \leq y(d)$ and $\widehat{c}_2 \leq y(\widehat{d})$ and the second inequality follows from Assumption A.5 that $z(d)+g_2(d_1)$ ($\equiv y(d)-g_1(d_2)$) is concave in d.
Case II. In this case, $\tilde{c}_1 = \lambda c_1 + (1-\lambda)\widehat{c}_1 + h_2$ and $\tilde{c}_2 = y(d^\lambda) - \tilde{c}_1$. By assumption, $\tilde{c}_1 \geq 0$. Furthermore, $\tilde{c}_2 = y(d^\lambda) - \tilde{c}_1 = y(d^\lambda) - (\lambda c_1 + (1-\lambda)\widehat{c}_1 + h_2) \geq y(d^\lambda) - (\lambda y(d) + (1-\lambda)y(\widehat{d})) - h_2 \geq 0$, where the final inequality follows because $y(d) - g_2(d_1)$ ($\equiv z(d) + g_1(d_2)$) is concave in d. Likewise, $\tilde{c}_2 - (\lambda c_2 + (1-\lambda)\widehat{c}_2) - h_1 = y(d^\lambda) - (\lambda(c_1+c_2) + (1-\lambda)(\widehat{c}_1+\widehat{c}_2)) - h_2 - h_1 \geq z(d^\lambda) - (\lambda z(d) + (1-\lambda)z(\widehat{d}) \geq 0$, where the first inequality follows from $c_1+c_2 \leq y(d)$ and $\widehat{c}_1+\widehat{c}_2 \leq y(\widehat{d})$. The final inequality follows from the concavity of $z(d)$, and is a strict inequality if $d \neq \widehat{d}$ because z is strictly concave.
Now consider the contract $(\tilde{x}(s^t), d^\lambda(s^t))_{t=0}^\infty$. We have, by the forgoing and by the concavity of u_i, that:

$$\begin{aligned} u_i(\tilde{c}_i - g_j(d_i^\lambda)) &\geq u_i(\lambda(c_i - g_j(d_i)) + (1-\lambda)(\widehat{c}_i - g_j(\widehat{d}_i))) \\ &\geq \lambda u_i(c_i - g_j(d_i)) + (1-\lambda)u_i(\widehat{c}_i - g_j(\widehat{d}_i)), \end{aligned} \tag{S.7}$$

where for one of the agents the first inequality is strict if $d \neq \widehat{d}$ (agent 1 in case I, agent 2 in case II). Thus, the contract $(\tilde{x}(s^t), d^\lambda(s^t))_{t=0}^\infty$ offers at least as much utility in each date-event pair as the average contract and is feasible and self-enforcing. By construction, utilities from the new contract are at least $\lambda(V_1, V_2^s(V_1)) + (1-\lambda)(\widehat{V}_1, V_2^s(\widehat{V}_1))$. Considering cases where $d \neq \widehat{d}$, the first claim of part (ii) follows by straightforward arguments.
Under Assumption A.4, it is only necessary to consider case II. Applying the same argument using the concavity of $z(d)$ shows that $(\tilde{x}(s^t), d^\lambda(s^t))_{t=0}^\infty$ offers at least as much utility in each date-event pair as the average contract and is feasible and self-enforcing. To establish the second claim of part (ii), for $d \neq \widehat{d}$ we again get a strict inequality because z is strictly concave; if u_i is strictly concave, then the second inequality in (S.7) is strict at some date, and hence, strict concavity of V_2 follows.
Using standard arguments, it then follows that $V_2^s(\cdot)$ is concave on an open interval $(\underline{V}_1, \bar{V}_1)$ where $\underline{V}_1$ and $\bar{V}_1$ are the respective infimum and supremum of the projection of the Pareto frontier onto agent 1's utilities. To show that $V_2^s(\cdot)$ is in fact concave, continuous and defined on $[\underline{V}_1, \bar{V}_1]$, consider a sequence $\{V_1^p\}_{p=1}^\infty \in (\underline{V}_1, \bar{V}_1)$, such that $V_1^p \downarrow \underline{V}_1$ (that is, from above). Since all variables belong to compact spaces, assume w.l.o.g. that a corresponding subsequence of optimal contracts $\{(x_t^p, d_t^p)_{t\geq 0})\}_{p=1}^\infty$ is convergent. Since all inequality constraints are weak, it is easily seen that this limit contract is self-enforcing. Therefore, an optimal contract must offer at least the utility to agent 2 from the limit contract, $V_2(\underline{V}_1) \geq \lim_{p\to\infty} V_2(V_1^p)$. Equally, it cannot offer more because this would violate concavity of the value function (creating a discontinuity at $\underline{V}_1$). The fact that $V_2(V_1)$ is continuous and decreasing on $(\underline{V}_1, \bar{V}_1)$ then implies that $V_2(\underline{V}_1) > V_2(V_1)$ for all $V_1 \in (\underline{V}_1, \bar{V}_1)$, and hence, that this point at $\underline{V}_1$ is constrained Pareto efficient. A similar argument applies at $\bar{V}_1$.

We now establish differentiability. Assume $\underline{V}_1^s < \bar{V}_1^s$ (this is established later). Fix $V_1 = V_1^o \in (\underline{V}_1^s, \bar{V}_1^s)$ and let the superscript "o" represent optimal values of other variables (these need not be unique). Since nothing depends on it, the notational dependence on the state s is dropped. Recall that by Proposition 1, output is positive. Hence there are three possibilities: (A) $c_1^o = 0$ and $c_2^o > 0$, (B) $c_1^o > 0$ and $c_2^o > 0$, or (C) $c_1^o > 0$ and $c_2^o = 0$ (only possibility B is relevant under Assumption A.4). First, consider case (B) and define $\Upsilon_1^o := V_1^o - d_2^o$ and $\Upsilon_2^o := V_2^o - d_1^o$ where $\Upsilon_1^o, \Upsilon_2^o \geq 0$ in any dynamic relational contract . Also $V_2^o = V_2(V_1^o)$. Likewise, we have the recursive equations $u_1(c_1^o - g_2(d_1^o)) + \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_1^{ro} = V_1^0$ and $u_2(z(d_1^o, d_2^o) + g_2(d_1^o) - c_1^0) + \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_2^{ro} = V_2^o$. Consider the following equations where the future values, but not (necessarily) the current values, are at their optimal levels:

$$V_1 - u_1(c_1 - g_2(d_1)) = \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_1^{ro},$$
$$V_2 - u_2(z(d_1, d_2) + g_2(d_1) - c_1)) = \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_2^{ro},$$
$$V_1 - d_2 = \Upsilon_1^o,$$
$$V_2 - d_1 = \Upsilon_2^o.$$

Since the functions u_i, g_i and z are continuous and differentiable, the *implicit function theorem* asserts the existence of continuous and differentiable functions $\tilde{c}_1(V_1)$, $\tilde{d}_1(V_1)$, $\tilde{d}_2(V_1)$ and $\tilde{V}_2(V_1)$ in an open interval around V_1^o such that $\tilde{c}_1(V_1^o) = c_1^o$ etc. and for each V_1 in the interval

$$V_1 - u_1(\tilde{c}_1(V_1) - g_2(\tilde{d}_1(V_1))) = \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_1^{ro},$$
$$\tilde{V}_2(V_1) - u_2(z(\tilde{d}_1(V_1), \tilde{d}_2(V_1)) + g_2(\tilde{d}_1(V_1)) - \tilde{c}_1(V_1)) = \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_2^{ro},$$
$$V_1 - \tilde{d}_2(V_1) = \Upsilon_1^o,$$
$$\tilde{V}_2(V_1) - \tilde{d}_1(V_1) = \Upsilon_2^o,$$

provided the determinant of the Jacobian matrix, J, of this system (all functions evaluated at the optimum V_1^o) satisfies:

$$|J| = u_1' \left(1 - u_2' \frac{\partial z}{\partial a_1} g_2'\right) \neq 0. \quad \text{(S.8)}$$

Given that $u_1' > 0$, the condition is equivalent to the linear independence constraint qualification, which holds unless $V_1 = \bar{V}_1^s$. We have $\tilde{V}_2(V_1^o) = V_2(V_1^o)$ and $\tilde{V}_2(V_1) \leq V_2(V_1)$ because $V_2(V_1)$ is an optimal value function. Since $V_2(V_1)$ is concave and given $\tilde{V}_2(V_1)$ is differentiable, and $\tilde{V}_2(V_1^o) = V_2(V_1^o)$ with $\tilde{V}_2(V_1) \leq V_2(V_1)$, Lemma 1 of Benveniste and Scheinkman (1979) can be applied, and therefore, it follows that $V_2(V_1)$ is differentiable at V_1^o.

Next, consider case (A): $c_1^o = 0$ and $c_2^o > 0$. We can proceed as before except that by Proposition 1(ii), agent 2's self-enforcing constraint is not binding. Thus, $V_2 > d_1$, and this constraint can be ignored. Instead, hold $c_1 = 0$ fixed. Therefore, consider small changes in the current contract (that is, varying a_1, a_2, V_1, V_2), which satisfy:

$$V_1 - u_1(-g_2^s(d_1)) = \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_1^{ro},$$
$$V_2 - u_2(z(d_1, d_2) + g_2(d_1))) = \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_2^{ro},$$
$$V_1 - d_2 = \Upsilon_1^o.$$

Here the implicit function theorem can be applied directly because the determinant of the Jacobian of the system is $u_1' g_2' > 0$. Again applying Lemma 1 of Benveniste and Scheinkman (1979) shows that the function $V_2(V_1)$ is differentiable at V_1. A similar argument applies to case (C). In particular, it can be shown that $V_2^{-1}(V_2)$ is differentiable at $V_2 = V_2(V_1)$. Since $V_2^{-1}(\cdot)$ is strictly decreasing, $V_2(\cdot)$ is differentiable at V_1. Since $V_2(V_1)$ is differentiable and is a concave function on $[\underline{V}_1^s, \bar{V}_1^s]$, it follows as a corollary to Theorem 24.1 in Rockafellar (1997) that the function has a continuous derivative.

Next we confirm that $\underline{V}_1^s < \bar{V}_1^s$. Suppose to the contrary that the frontier consists of a single point. We establish a contradiction. For cases (A), (B) and (C) let $|J|$ denote the determinant of the Jacobian of the systems described above and let $|J'|$ be the corresponding determinant of the Jacobian with the agent indices swapped. If either $|J| \neq 0$ *or* $|J'| \neq 0$, then the existence of the differentiable subfunction $\tilde{V}_2(V_1)$ establishes that there are feasible points which offer one of the agents a higher utility, contradicting the hypothesis. Thus, it can only be that case (B) applies and that $|J| = |J'| = 0$. Rewriting the term in brackets in (S.8), this implies

$$(1 - u_j'((\partial z)/(\partial a_i))g_j') = (u_j'/D_j')((\partial \phi_j/\partial a_i) + 1 - (\partial y/\partial a_i)) = 0, \tag{S.9}$$

$i \neq j$, $i,j = 1,2$. Since $(u_j'/D_j') > 0$, this implies $(\partial y/\partial a_i) - 1 - (\partial \phi_j/\partial a_i) = 0$. Consider a small increase in a_1 of Δ. Suppose that of the increase in output, agent 2 receives the increase in her default, approximately $\Delta\partial\phi_2/\partial a_1$, while the remainder is allocated to agent 1. Given equation (S.9), the remainder is approximately $\Delta\partial y/\partial a_1 - \Delta\partial\phi_2/\partial a_1 = \Delta$. Since Δ is agent 1's extra effort cost, she suffers no more than a second-order loss, while agent 2 has a first-order gain in utility of approximately $u_2'\Delta\partial\phi_2/\partial a_1$. When state s recurs, make a corresponding increase in a_2, and thereafter continue alternating between the two agents. Since $\delta > 0$, for Δ small enough, there is a first-order gain in discounted utilities for both agents, and the first-order increase in deviation utilities is always more than matched by an increase in the constructed contract utilities. Thus, an allocation which offers strictly more than the initial equilibrium can be supported as an equilibrium, again giving a contradiction. We now show that

$$V_2^{(+)}(\underline{V}_1) = 0 \quad \text{and} \quad V_2^{(-)}(\bar{V}_1) = -\infty.$$

Suppose to the contrary of the assertion that $V_2^{(+)}(\underline{V}_1) < 0$ (it cannot be positive by definition of it being a Pareto frontier). By a previous argument, for every $V_1 \in [\underline{V}_1, \bar{V}_1]$ at which the implicit function theorem can be applied, there is a continuously differentiable function $\tilde{V}_2(V_1)$ which describes utilities to agent 2 from dynamic relational contracts which yield V_1 to agent 1. Moreover, the theorem together with the optimality of the function $V_2(V_1)$ imply that there is an open neighborhood of V_1 such that $V_2(V_1 + \epsilon) \geq \tilde{V}_2(V_1 + \epsilon)$ for all $\epsilon \geq 0$. At $V_1 = \underline{V}_1$ this therefore implies $\tilde{V}_2'(\underline{V}_1) \leq V_2^{(+)}(\underline{V}_1) < 0$. The implicit function theorem (which applies if $\tilde{V}_2'(V_1)$ is finite) then implies that there is an $\epsilon > 0$ with $\tilde{V}_2(V_1 - \epsilon) > \tilde{V}_2(\underline{V}_1) = V_2(\underline{V}_1)$ corresponding to a dynamic relational contract . Consequently, the Pareto frontier must extend below $\underline{V}_1$, which is a contradiction. A similar argument applies at $\bar{V}_1$. ■

Lemma 5. Under Assumptions 1-3 and under either Assumption A.4 or Assumption A.5, $d_i^s(V_1)$ is a continuous function of V_1 for each $s \in \mathcal{S}$ and $i = 1,2$.

Proof. Again we suppress the notational dependence on the state s. To see the uniqueness of d as a function of V_1, suppose to the contrary that $(x, d, (V_1^r)_{r\in\mathcal{S}})$ and $(\widehat{x}, \widehat{d}, (\widehat{V}_1^r)_{r\in\mathcal{S}})$ are both optimal at V_1 with $d \neq \widehat{d}$. Consider the convex combinations $d^\lambda = \lambda d + (1-\lambda)\widehat{d}$ for some $\lambda \in (0,1)$ and define $\tilde{c}$ and cases I and II as in the proof of Proposition 2. In case I, agent 1 is strictly better off while agent 2 is no worse off; in case II, as $d \neq \widehat{d}$, agent 2 is strictly better off while agent 1 is no worse off. As shown in the proof of Proposition 2, this change is feasible, and delivers a payoff profile that is Pareto-superior to that at V_1, contrary to the optimality of the two original contracts.
To establish continuity of $d_i(V_1)$, let $\{V_1^p\}_{p=1}^\infty$ be such that $V_1^p \to V_1^*$. Since all variables belong to compact spaces, assume w.l.o.g. that the corresponding sequence of optimal choices $\{(x^p, d^p, (V_1^{rp})_{r\in\mathcal{S}})\}_{p=1}^\infty$ is convergent to $(\bar{x}, \bar{d}, (\bar{V}_1^r)_{r\in\mathcal{S}})$, and let $(x^*, d^*, (V_1^{r*})_{r\in\mathcal{S}})$ be the optimal choices at V_1^*. Assume that $\bar{d} \neq d^*$; a contradiction will be established. By continuity of all the constraints, $(\bar{c}, \bar{d}, (\bar{V}_1^r)_{r\in\mathcal{S}})$ is feasible for V_1^*, and since $V_2^s(V_1)$ is continuous (from Proposition 2), this choice attains the maximum, contradicting the uniqueness of d as a function of V_1^s. ■

Proposition 3. $V_2^s(V_1)$, $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$, is a solution to the following program

$$\text{[P1]} \qquad V_2^s(V_1) = \max_{a \geq 0, x \geq \underline{x}, (V_1^r \in \mathbb{R})_{r \in \mathcal{S}}} \left\{ u_2(x_2) + \delta \sum\nolimits_{r \in \mathcal{S}} \pi_{sr} V_2^r(V_1^r) \right\}$$

subject to

$$\text{(S.10a)} \qquad u_1(x_1) + \delta \sum\nolimits_{r \in \mathcal{S}} \pi_{sr} V_1^r \geq V_1 : \qquad \lambda$$

$$\text{(S.10b)} \qquad V_1 \geq D_1^s(a_2) : \qquad \mu_1$$

$$\text{(S.10c)} \qquad u_2(x_2) + \delta \sum\nolimits_{r \in \mathcal{S}} \pi_{sr} V_2^r(V_1^r) \geq D_2^s(a_1) : \qquad \mu_2$$

$$\text{(S.10d)} \qquad V_1^r \geq \underline{V}_1^r : \qquad \delta \pi_{sr} \nu_1^r$$

$$\text{(S.10e)} \qquad V_1^r \leq \bar{V}_1^r : \qquad \delta \pi_{sr} \nu_2^r$$

$$\text{(S.10f)} \qquad x_i + a_i \geq 0 : \quad i, j = 1, 2, \quad i \neq j \qquad \gamma_i$$

$$\text{(S.10g)} \qquad x_1 + x_2 \leq z^s(a_1, a_2) : \qquad \psi$$

Proof. Arguments follow standard lines. Briefly: take an optimal contract corresponding to $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$ in state s (from Lemma 4 such a contract exists) with its initial values for a, x, and $(V_i^r)_{r \in \mathcal{S}}$, $i = 1, 2$, being continuation payoffs. First, (S.10d) – (S.10e) must hold, since, otherwise a Pareto-dominating continuation is feasible, which also relaxes all constraints and increases current payoffs. By definition of a dynamic relational contract, (S.10b) – (S.10c) must hold (the latter since $V_2^r \leq V_2^r(V_1^r)$, by definition of $V_2^r(\cdot)$, so, if it were violated, agent 2 would be better off deviating) and also (S.10f) – (S.10g) must be satisfied for feasibility. Conversely, a, x, and $(V_1^r)_{r \in \mathcal{S}}$ satisfying (S.10a) – (S.10g) correspond to a dynamic relational contract with (a, x) in the initial period, and the optimal contract corresponding to each V_1^r as the continuation contract in state r. If the maximum in [P1] is not attained by an optimal contract, then replacing it by a set of choices with a higher value of the maximand will lead to a Pareto-superior dynamic relational contract, contradicting optimality. ■

Lemma 6. Under Assumptions 1-3, and for $i, j = 1, 2$, $i \neq j$, for any history s^t, (i) if $V_i(s^t) > d_j(s^t)$, then $a_j(s^t) \geq a_j^*(a_i(s^t), s_t)$; (ii) if $c_i(s^t) > 0$, then $a_i(s^t) \leq a_i^*(a_j(s^t), s_t)$.

Proof. Again we drop the state notation for the proof. Parts (i) and (ii) follow directly from the first-order condition (3.2c). (i) If $V_i > d_j$, then $\mu_i = 0$ and therefore, from (3.2c), $\partial z^s(a)/\partial a_j \leq 0$. Thus, $a_j \geq a_j^*(a_i, s)$, that is, there is no underinvestment. (ii) Equally, if $c_i > 0$, then $\gamma_i = 0$ and therefore, from (3.2c), $\partial z^s(a)/\partial a_i \geq 0$. Thus, $a_i \leq a_i^*(a_j, s)$, that is, there is no overinvestment. ■

Computation of the illustrative example from Section 1

In this section we show how to compute the solution to the illustrative example presented in the Introduction of the paper. The example has an additive production technology

$$y(a_1, a_2) = f_1(a_1) + f_2(a_2) = 2\left(\sqrt{a_1} + \sqrt{a_2}\right).$$

Preferences are common and exhibit constant absolute risk aversion with parameter 1/2:

$$u_i(x) = 2\left(1 - \mathrm{e}^{-\frac{1}{2}x}\right).$$

The conditionally efficient actions are independent of the other agent's action and given by $a_1^* = a_2^* = 1$ and the first-best surplus $z(a^*) = 2$. The breakdown payoffs are of the form $\phi_i(a_1, a_2) = \theta_{i1} f_1(a_1) + \theta_{i2} f_2(a_2)$ where the parameters are $\theta_{11} = \theta_{22} = 0$ and $\theta_{12} = \theta_{21} = 1$. Given the additive technology, the Nash best-response functions are dominant strategies, and hence, $a_i^N = a_i^{NE} = 0$.[26] With these assumptions, the deviation payoffs are:

$$D_i(a_j) = u_i\left(2\sqrt{a_j}\right) = 2\left(1 - \mathrm{e}^{-\sqrt{a_j}}\right).$$

At the first-best, the net consumption of both agents is $x_i^* = 1$. This is sustainable provided

$$\frac{u_i(1)}{(1-\delta)} = \frac{2(1-\mathrm{e}^{-\frac{1}{2}})}{(1-\delta)} \geq D_i(1) = 2(1-\mathrm{e}^{-1}),$$

or $\delta \geq (1+\sqrt{\mathrm{e}})^{-1}$. Here we consider the knife-edge case $\delta = (1+\sqrt{\mathrm{e}})^{-1}$, in which case the first best is just sustainable for an equal split of the surplus.

Since the example is symmetric, we first compute the solution for low values of $V_1 < u_1(2)$ and the case for higher values of V_1 is completely symmetric. Consider a low value of $V_1 \geq \underline{V}_1$ where agent 1 is constrained, that is $V_1 < u_1(2) = 2(1+\mathrm{e}^{-1})$. By construction, agent 2 is unconstrained and the optimal action for agent 1 is the conditionally efficient action. The action of agent 1, and hence surplus is determined by the binding constraint $V_1 = D_1(a_2)$. In particular, for $V_1 \in [\underline{V}_1, u_1(2)]$

$$\begin{aligned}
a_1(V_1) &= 1,\\
a_2(V_1) &= \left(\tfrac{1}{2}u_1^{-1}(V_1)\right)^2 = \left(\log\left(1-\tfrac{V_1}{2}\right)\right)^2,\\
z(V_1) &= 1 + u^{-1}(V_1) - \left(\tfrac{1}{2}u_1^{-1}(V_1)\right)^2 = 1 - 2\log\left(1-\tfrac{V_1}{2}\right) - \left(\log\left(1-\tfrac{V_1}{2}\right)\right)^2,\\
z'(V_1) &= u_1^{-1\prime}(V_1)\left(1+u^{-1}(V_1)\right) = \left(1+\log\left(1-\tfrac{V_1}{2}\right)\right)\left(1-\tfrac{V_1}{2}\right)^{-1}.
\end{aligned}$$

It is easily checked that the surplus function $z(V_1)$ is increasing and concave in this region with $z(0) = z'(0) = 1$ and $z(V_1) = 2$ and $z'(V_1) = 0$ when $V_1 = u(2)$. It follows, (from equation (4.1) in the paper), that $\sigma^+(V_1) \geq \sigma(V_1)$ with equality only holding when $V_1 = u(2)$ where, because agents are symmetric, $\sigma^+(V_1) = \sigma(V_1) = 1$.

If the Pareto frontier were known, in particular, if $\sigma(V_1)$ were known, then, for a given V_1, the net consumption of each agent can be determined from equation (4.1) together with the first-order condition $\sigma^+(V_1) = u_2'/u_1'$ and that the sum of net consumptions equals the current surplus. Since the Pareto-frontier is not known, equation (4.1) can be used to determine the net consumption of agent 1, say, as a function of V_1 and σ. The equation for net consumption x_1 is determined by

$$\mathrm{e}^{x_1}\mathrm{e}^{-\frac{1}{2}z(V_1)} - \sigma = \mathrm{e}^{\frac{1}{2}x_1}\mathrm{e}^{-\frac{1}{2}z(V_1)}z'(V_1),$$

which can be solved explicitly to give

$$x_1(V_1,\sigma) = \log\left(\tfrac{1}{2}\left((z'(V_1))^2 + 2\sigma\mathrm{e}^{\frac{1}{2}z(V_1)} + z'(V_1)\sqrt{(z'(V_1))^2 + 4\sigma\mathrm{e}^{\frac{1}{2}z(V_1)}}\right)\right).$$

For the given value of V_1 and σ, the promise-keeping constraint then determines $V_1^+(V_1,\sigma) = (1/\delta)(V_1 - u(x_1(V_1,\sigma)))$. It can be checked that $x_1(V_1,\sigma)$ is increasing in σ (so that $V_1^+(V_1,\sigma)$ is decreasing in σ) and that $V_1^+(V_1,\sigma) - V_1$ is increasing in V_1. Hence, the locus of values where $V_1^+(V_1,\sigma) = V_1$ is upward sloping and furthermore, it passes through $(0,0)$ and $(u(2),1)$. $V_1^+(V,\sigma) > V_1$ below this locus and $V_1^+(V_1,\sigma) < V_1$ above it. The locus of points where $\sigma^+(V_1,\sigma) = \sigma$ occurs when $z'(V_1) = 0$, that is for $V_1 = u_1(2)$. The

[26] Strictly, this violates our condition that Nash actions are strictly positive. However, the importance of the assumption of positive Nash equilibrium actions was to rule out trivial contracts. Trivial contracts with zero actions are not optimal in this example for the parameter values chosen, and hence, the substance of the theorems does apply.

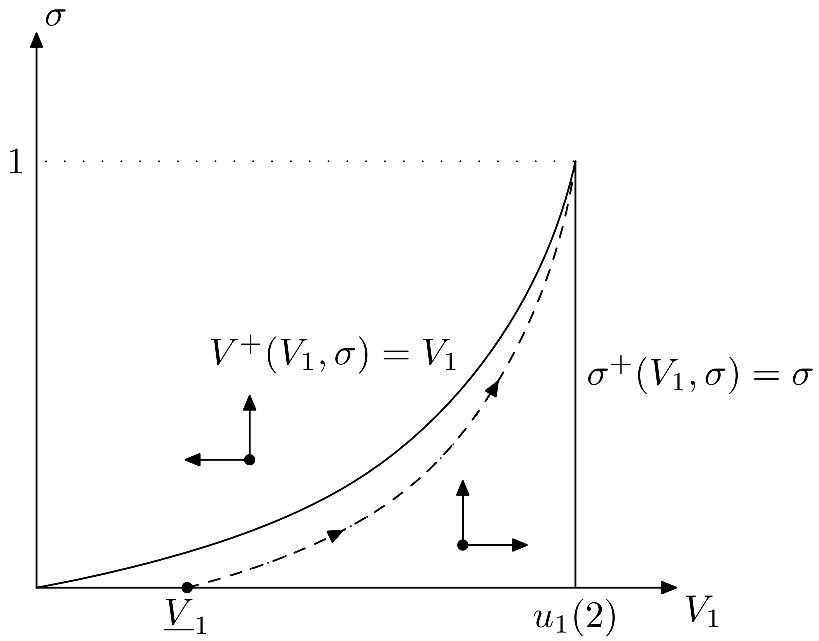

Figure S.1: Phase Diagram

phase diagram for the change in V_1 and σ is illustrated in Figure S.1. The optimal solution where agent 1 is constrained corresponds to the unique stable saddle path in (V,σ)-space that converges to the first-best allocation $(V,\sigma) = (u(2), 1)$. The complete solution can be numerically approximated by starting with $\sigma = 0$, and finding the value of $V_1 = \underline{V}_1$ on the saddle path. The Pareto-frontier is found by calculating the expected discounted utility of agent 2 along the computed saddle path. In the numerical calculations this is done by using ten iterations and then assuming the first-best allocation is attained. The approximation gives $\underline{V}_1 \approx 0.353328$ and $\bar{V}_1 = V_2(\underline{V}_1) \approx 1.53803$.

Since the example is symmetric, the solution when V_1 is large enough such that agent 2 is constrained is computed symmetrically. The solution for the Pareto-frontier, surplus and actions of both agents is illustrated in Figure S.2. The point "S" indicates the first-best allocation.

Proofs of lemmas and propositions for Section 4

For all proofs in this subsection, we maintain Assumptions 1-3 and A.4. Additionally it is assumed that agents are risk averse, that is, u_i is strictly concave for $i = 1, 2$.

Lemma 7. For each $s \in \mathcal{S}$, a solution to [P1] has the property that $z^s(a_1, a_2)$ is maximized over $a \in \mathbb{R}^2_+$ subject to $V_1 \geq D^s_1(a_2)$ and $V^s_2(V_1) \geq D^s_2(a_1)$.

Proof. We work in terms of the variables d rather than directly in terms of the actions a and show that $z^s(d_1, d_2)$ is maximized subject to $V_1 \geq d_2$ and $V^s_2(V_1) \geq d_1$. Suppose otherwise, and replace $(d_1(V_1), d_2(V_1))$ by some $(d_1, d_2) \in \mathcal{D}(s)$ satisfying these constraints with $z^s(d_1, d_2) > z^s(d_1(V_1), d_2(V_1))$. In doing so, hold $c_1 - g^s_2(d_1)$ and $(V^r_1)_{r\in\mathcal{S}}$ constant. With these changes, all constraints are satisfied, but the maximand is increased, leading to a contradiction. ■

Lemma 8. For each $s \in \mathcal{S}$, the surplus function $z^s(V_1)$ is continuous, concave and differentiable in V_1.

Proof. Taking each property in turn.
Continuity: Continuity follows straightforwardly from the Theorem of the Maximum.
Concavity: Take any V_1 and V'_1 in $[\underline{V}^s_1, \bar{V}^s_1]$ and the convex combination $V^\lambda_1 = \lambda V_1 + (1-\lambda)V'_1$, $0 \leq \lambda \leq 1$.

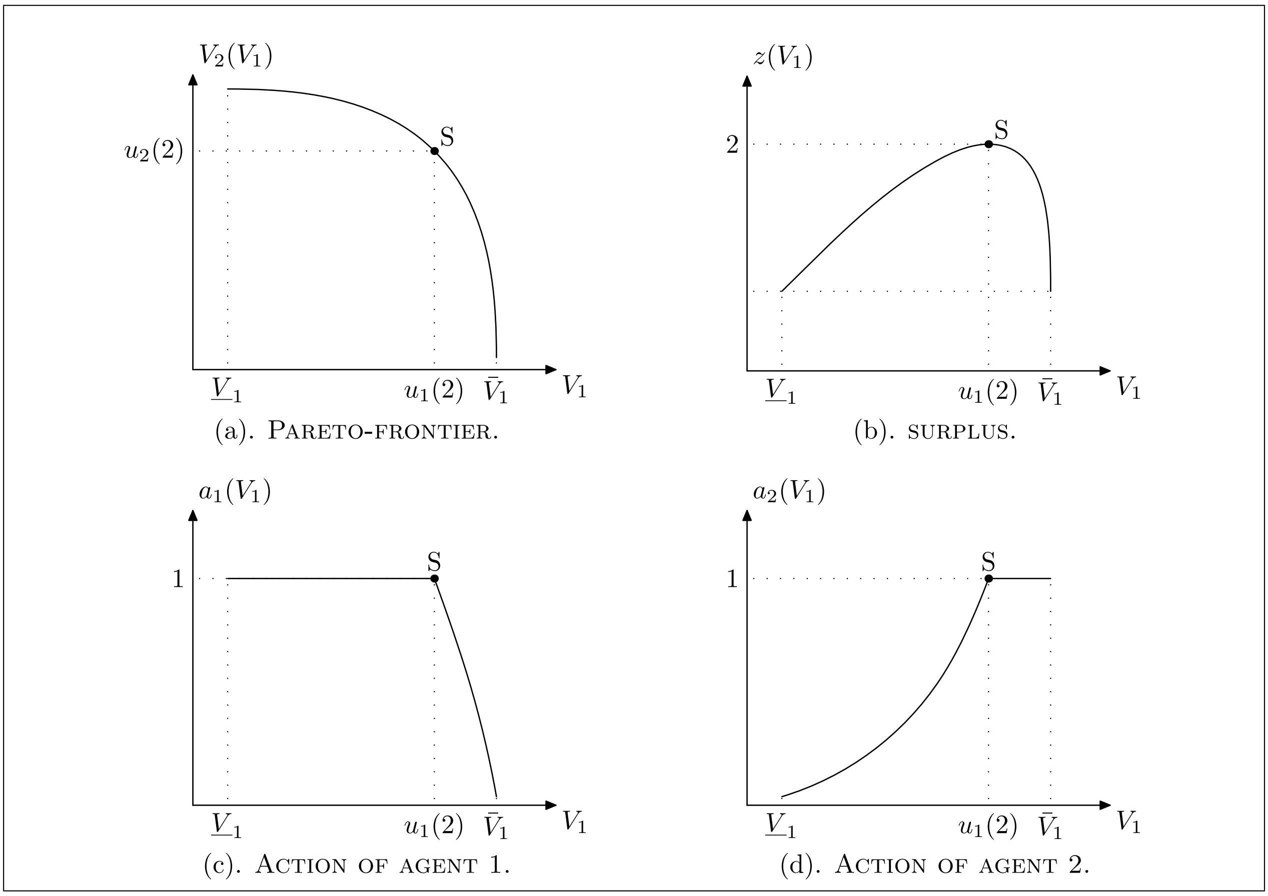

Figure S.2: Example where $u_i(x) = 2(1 - \exp(-\frac{1}{2}x))$, $f(a) = 2(\sqrt{a_1} + \sqrt{a_2})$ and deviation payoff $D_i(a_j) = 2(1 - \exp(-\sqrt{a_j}))$. Point "S" is the stationary point.

Let $d_i^\lambda := \lambda d_i(V_1) + (1-\lambda) d_i(V_1')$. Since $V_i \geq d_j(V_1)$ and $V_i' \geq d_j(V_1')$, it follows that $V_1^\lambda \geq d_2^\lambda$. Similarly, $V_2^s(V_1^\lambda) \geq d_1^\lambda$, from the concavity of $V_2^s(V_1)$. Consequently, $(d_1^\lambda, d_2^\lambda)$ is feasible at V_1^λ, and therefore, by Lemma 7 and the concavity of $z^s(d_1, d_2)$, $z^s(d_1(V_1^\lambda), d_2(V_1^\lambda)) \geq z^s(d_1^\lambda, d_2^\lambda) \geq \lambda z^s(d_1(V_1), d_2(V_1)) + (1 - \lambda) z^s(d_1(V_1'), d_2(V_1'))$. Thus, the concavity of $z^s(V_1)$ is established.

Differentiability: To establish differentiability fix $\hat{V}_1 \in (\underline{V}_1^s, \bar{V}_1^s)$ with optimal choices $d_1(\hat{V}_1)$ and $d_2(\hat{V}_1)$, and consider a V_1 in a neighborhood of $\hat{V}_1$ ($\subset [\underline{V}_1^s, \bar{V}_1^s]$). Consider $(\tilde{d}_1(V_1), \tilde{d}_2(V_1))$ satisfying $\tilde{d}_2(V_1) - d_2(\hat{V}_1) = V_1 - \hat{V}_1$ and $\tilde{d}_1(V_1) - d_1(\hat{V}_1) = V_2^s(V_1) - V_2^s(\hat{V}_1)$. By construction, $(\tilde{d}_1(V_1), \tilde{d}_2(V_1))$ is feasible, and by the differentiability of $V_2^s(V_1)$ and $z^s(d_1, d_2)$, this traces out a differentiable function for V_1, $z^s(\tilde{d}_1(V_1), \tilde{d}_2(V_1))$ in the neighborhood of $\hat{V}_1$, with $z^s(\tilde{d}_1(\hat{V}_1), \tilde{d}_2(\hat{V}_1)) = z^s(\hat{V}_1)$. From Part A, $z^s(\tilde{d}_1(V_1), \tilde{d}_2(V_1)) \leq z^s(V_1)$. Therefore, applying Lemma 1 of Benveniste and Scheinkman (1979) establishes differentiability. ■

Lemma 9. For each $s \in \mathcal{S}$ (i) $\mathrm{d}z^s(V_1)/\mathrm{d}V_1 > 0$ (< 0) implies $\mu_1^s(V_1) > 0$ ($\mu_2^s(V_1) > 0$); (ii) there are two critical values $\bar{\chi}_1^s \in (\underline{V}_1^s, \bar{V}_1^s]$ and $\underline{\chi}_1^s \in [\underline{V}_1^s, \bar{V}_1^s)$, such that $d_2^s(V_1) = V_1$ for all $V_1 \leq \bar{\chi}_1^s$ and $d_1^s(V_1) = V_2^s(V_1)$ for all $V_1 \geq \underline{\chi}_1^s$. Moreover, $\mu_1^s(V_1) = 0$ for $\bar{V}_1^s > V_1 \geq \bar{\chi}_1^s$ and $\mu_2^s(V_1) = 0$ for $\underline{V}_1^s < V_1 \leq \underline{\chi}_1^s$ (if such V_1 exist). If the efficient actions can be sustained in state s, then $\bar{\chi}_1^s \leq \underline{\chi}_1^s$. Otherwise, $\bar{\chi}_1^s > \underline{\chi}_1^s$, and surplus is maximized for a unique value of $V_1 \in (\underline{\chi}_1^s, \bar{\chi}_1^s)$ at which both constraints bind.

Proof. Since nothing depends on it, we drop the state superscript in what follows. Result (i) follows immediately from (4.1) (which follows from the first-order conditions) and the first-order condition (3.2a) (setting $\nu_i^r = 0$ for $i = 1, 2$).

To prove result (ii), let the surplus function be at a maximum between $\underline{V}_1^*$ and $\bar{V}_1^*$ (with a unique maximum if $\underline{V}_1^* = \bar{V}_1^*$). There are two possibilities: case (a), $\underline{V}_1^* < \bar{V}_1^*$, and case (b), $\underline{V}_1^* = \bar{V}_1^* =: \hat{\chi}_1$. In case (a), it follows from (i) that $\mu_1(V_1) > 0$ for $V_1 < \underline{V}_1^*$, and hence, $d_1(V_1) = V_2(V_1)$ for $V_1 \leq \underline{V}_1^*$, where the weak inequality follows by continuity of $d_1(V_1)$ and $V_2(V_1)$. Likewise, $\mu_2(V_1) > 0$ for $V_1 > \bar{V}_1^*$, and $d_2(V_1) = V_1$ for $V_1 \geq \bar{V}_1^*$. Since $z(d)$ is strictly concave, and $z(V_1)$ is constant on $[\underline{V}_1^*, \bar{V}_1^*]$, it follows (see proof of Lemma 14) that actions are first-best and $\mu_1(V_1) = \mu_2(V_1) = 0$ on this interval. Next, consider some $\hat{V}_1 < \underline{V}_1^*$. We want to show that $\mu_2(\hat{V}_1) = 0$. Suppose to the contrary that $\mu_2(\hat{V}_1) > 0$. Then, setting $\gamma_i = 0$, $i = 1, 2$ in the first-order condition (3.2c) shows that $\partial z(d_1(\hat{V}_1), d_2(\hat{V}_1))/\partial d_1 > 0$ and $d_1(\hat{V}_1) = V_2(\hat{V}_1) > V_2(\underline{V}_1^*) \geq d_1(\underline{V}_1^*)$, where the strict inequality follows because $V_2(\cdot)$ decreasing. Since $\mu_2(\underline{V}_1^*) = 0$ (by efficiency), it follows that $\partial z(d_1(\underline{V}_1^*), d_2(\underline{V}_1^*))/\partial d_1 = 0$. Since $d_1(\hat{V}_1) > d_1(\underline{V}_1^*)$, and because by assumption $\partial^2 z/\partial d_1^2 < 0$ and $\partial^2 z/\partial d_1 \partial d_2 \geq 0$, it follows that $d_2(\hat{V}_1) > d_2(\underline{V}_1^*)$. But $\mu_1(V_1) > 0$ for $V_1 < \underline{V}_1^*$ by the first part of the proof, so that $d_2(\hat{V}_1) = \hat{V}_1 < \underline{V}_1^* = d_2(\underline{V}_1^*)$ (where the last equality follows by continuity), yielding a contradiction. A similar argument shows that $\mu_1(V_1) = 0$ for $V_1 \geq \bar{V}_1^*$.
For case (b), define $\tilde{V}_1^{\mu_2} \in [\underline{V}_1, \hat{\chi}_1]$ to be the largest (supremum) value of V_1 with $\mu_2(V_1) = 0$ (recall $\mu_2(V_1) > 0$ for $V_1 > \hat{\chi}_1$ by part (i)). First, suppose such a value exists. Noting that $\mu_1(V_1) > 0$ for $V_1 < \hat{\chi}_1$, for $\hat{V}_1 < V_1^{\mu_2}$, replace $\underline{V}_1^*$ by $V_1^{\mu_2}$ in the argument given in case (a), to show that $\mu_2(\hat{V}_1) = 0$ for $\hat{V}_1 < V_1^{\mu_2}$. A symmetric argument can be used to show $\mu_1(\hat{V}) = 0$ for all $\hat{V} > V_1^{\mu_1}$, if there exists a $V_1^{\mu_1} \in [\hat{\chi}_1, \bar{V}_1]$ such that $\mu_1(V_1^{\mu_1}) = 0$. then $\mu_1(\hat{V}) = 0$ for all $\hat{V} > V_1^{\mu_1}$. Therefore, set $\underline{\chi}_1 = \tilde{V}_1^{\mu_2}$ and $\bar{\chi}_1 = \tilde{V}_1^{\mu_1}$. Note also, that if $\bar{\chi}_1^s = \underline{\chi}_1^s$ $(= \hat{\chi}_1)$, then $\mu_1(\hat{\chi}_1) = \mu_2(\hat{\chi}_1) = 0$, by continuity of the multipliers in V_1 (setting $\gamma_i = 0$, $i = 1, 2$ in equation (3.2c) and modifying the argument in Lemma 14), then actions are first-best at $\hat{\chi}_1$. Finally, if there is no $\tilde{V}_1^{\mu_2} \in [\underline{V}_1, \hat{\chi}_1]$ to be the largest (supremum) value of V_1 with $\mu_2(V_1) = 0$, then set $\underline{\chi}_1 = \underline{V}_1$. In this case, $\mu_2(\underline{V}_1) > 0$ and both actions are under-efficient at $\underline{V}_1$. Likewise, if there is no $\tilde{V}_1^{\mu_1}$ such that $\mu_1(\tilde{V}_1^{\mu_1}) = 0$, then $\bar{\chi}_1 = \bar{V}_1$, $\mu_1(\bar{V}_1) > 0$ and both actions are under-efficient at $V_1 = \bar{V}_1$. ■

Proposition 4. With risk-averse agents and under Assumption A.4 (i) there is no overinvestment, $\partial z^s(a(s^t))/\partial a_i \geq 0$, $i = 1, 2$, all s^t; (ii) surplus $z^s(V_1)$ is a concave differentiable function (strictly concave if $s \in \mathcal{S}_*^c$) with maximum at unique CSM actions; (iii) at V_1 such that $z^s(V_1)$ is maximized, either efficient actions $a^*(s)$ are sustainable (by definition, if $s \in \mathcal{S}_*$) or *both* constraints bind (if $s \in \mathcal{S}_*^c$) and $a^s(V_1) < a^*(s)$; (iv) for each $V_1 \in [\underline{V}_1^s, \bar{V}_1^s]$, $\sigma_s^+(V_1)$, the (absolute value of the) common slope of the Pareto-frontiers next period, and $\sigma_s(V_1)$, the slope of the current Pareto-frontier, satisfy

$$\sigma_s^+(V_1) - \sigma_s(V_1) = u_2' \frac{\mathrm{d}z^s(V_1)}{\mathrm{d}V_1}.$$

Proof. (i) By Proposition 2(ii), the Pareto-frontier is strictly concave. This implies not only is $a_i^s(\cdot)$ a continuous function of V_1 but it also can be established (by adapting the proof of Lemma 5, using the strict concavity of the Pareto-frontiers and the strict concavity of the utility functions) that $x_i^s(\cdot)$ and $V_1^{s,r}(\cdot)$ are continuous functions of V_1. Our previous discussion of [P1] has shown that the endpoint constraints (3.1d) and (3.1e) do not bind. Equally, by part (b) of Assumption A.4, $x_i > 0$ in an optimal contract and hence, since actions are non-negative, the consumption constraints (3.1f) do not bind. Since constraints (3.1f) do not bind, setting $\gamma_i = 0$, $i = 1, 2$, in equation (3.2c) shows that $\partial z^s/\partial a_i \geq 0$, $i = 1, 2$. This implies, $a_i^s(V_1) \leq a_i^s(a_j(V_1), s)$, $i, j = 1, 2$, $i \neq j$. (ii) follows from Lemma 5, and the fact that for $s \in \mathcal{S}_*^c$, $\arg\max_{V_1} z^s(V_1)$ is unique and so the CSM action is unique, while for $s \in \mathcal{S}_*$ it is $a^*(s)$ by definition. (iii) follows from Lemma 9: if $s \in \mathcal{S}_*^c$ then $\bar{\chi}_1^s > \underline{\chi}_1^s$, so both constraints bind. By Lemma 7 $z^s(a_1, a_2)$ is maximized over $a \in \mathbb{R}_+^2$ subject to $V_1 \geq D_1^s(a_2)$ and $V_2^s(V_1) \geq D_2^s(a_1)$, and so $a_i \leq a_i^*(a_j)$, $i = 1, 2$ (otherwise surplus could be increased by cutting a_i). Hence by $a_i^*()$ non-decreasing and $a \neq a^*$ the result follows. (iv) Setting $\nu_i^r = 0$ for $i = 1, 2$ in equation (3.2a), and also setting $\gamma_i = 0$ for $i = 1, 2$ in equation (3.2b), and substituting gives:

$$\text{(S.11)} \qquad \sigma_s^+(V_1) - \sigma_s(V_1) = u_2' \left(-\sigma_s(V_1) \frac{\partial z^s(d_1^s(V_1), d_2^s(V_1))}{\partial d_1} + \frac{\partial z^s(d_1^s(V_1), d_2^s(V_1))}{\partial d_2} \right).$$

The bracketed term on the right-hand-side of equation (S.11) is equal to $\mathrm{d}z^s(V_1)/\mathrm{d}V_1$. To see this recall that $\partial z^s/\partial d_i \geq 0$ and first note that if $\partial z^s/\partial d_1 > 0$, then $\mu_2 > 0$ and (3.1c) holds as an equality. Therefore, $d_1^s(V_1) = V_2^s(V_1)$ and consequently $\mathrm{d}d_1^s/\mathrm{d}V_1 = V_2^{s\prime}(V_1) = -\sigma_s(V_1) < 0$. Hence, $-\sigma_s(V_1)(\partial z^s/\partial d_1)$ equals $(\partial z^s/\partial d_1)(\mathrm{d}d_1^s/\mathrm{d}V_1)$, with the same equality trivially holding if $\partial z^s/\partial d_1 = 0$. Likewise, if $\partial z^s/\partial d_2 > 0$, then (3.1b) holds an equality and $\mathrm{d}d_2^s/\mathrm{d}V_1 = 1$, and hence, $(\partial z^s/\partial d_2) = (\mathrm{d}d_2^s/\mathrm{d}V_1)(\partial z^s/\partial d_2)$ (again, also holding trivially if $\partial z^s/\partial d_2 = 0$). From Lemma 8, the surplus function is differentiable and therefore, using the total derivative of $z^s(d_1^s(V_1), d_2^s(V_1))$ with respect to V_1, it follows that bracketed term in equation (S.11) is equal to $\mathrm{d}z^s(V_1)/\mathrm{d}V_1$. Hence, equation (S.11) can be written as

$$\sigma_s^+(V_1) - \sigma_s(V_1) = u_2' \frac{\mathrm{d}z^s(V_1)}{\mathrm{d}V_1},$$

which is equation (4.1) in the text. ■

Proof of proposition 5 in Section 4

Proposition 5. For each state $s \in \mathcal{S}$, (i) for all $\rho \in \mathbb{R}_+$, $h(\rho, s) \to h^{RS}(\rho, s)$ as $\theta_{ij} \to 0$, $i, j = 1, 2$, $i \neq j$, all s. (ii) If optimal contracts in the risk-sharing problem improve upon autarky, then for any η satisfying $(1/2)(\bar{\rho}_s^{RS} - \underline{\rho}_s^{RS}) > \eta > 0$, all s, there exists $\epsilon > 0$ such that for $\theta_{ij}^s < \epsilon$, $i, j = 1, 2$, $i \neq j$, all s, $h(\rho, s) = \rho$ for all $\rho \in [\underline{\rho}_s^{RS} + \eta, \bar{\rho}_s^{RS} - \eta]$.

Proof. To nest the hold-up and risk-sharing models consider the following problem:

$$\text{[P2]} \qquad \max_{(a(s^t) \geq 0, x(s^t) \geq \underline{x})_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t u_2(x_2(s^t)) \mid s_0\right]$$

subject to

$$\text{(S.12a)} \qquad \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t u_1(x_1(s^t)) \mid s_0\right] \geq V_1(s_0)$$

$$\text{(S.12b)} \qquad u_i(x_i(s^t)) + \mathbb{E}\left[\sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_i(x_i(s^\tau)) \mid s^t\right] \geq F_i^s(a_j(s^t), \theta, \xi) \quad i = 1, 2 \text{ and } \forall s^t$$

$$\text{(S.12c)} \qquad x_1(s^t) + x_2(s^t) \leq \hat{z}^s(a(s^t)) \quad \forall s^t,$$

where we define $\theta \equiv (\theta_{ij}^s)_{i,j=1,2,s \in \mathcal{S}}$. For $F_i^s(a_j, \theta, \xi) = D_i^s(a_j; \theta)$ (where the dependence of the deviation utility on the default parameters has been made explicit) and $\hat{z}^s(a) = z^s(a) := f_1^s(a_1) + f_2^s(a_2) - a_1 - a_2$, the solution to this problem is the solution to our hold-up problem (henceforth, the HU problem). Define $Y_i^s := f_i^s(a_i^*(s)) - a_i^*(s)$, $\hat{z}^s(a) = Y_1^s + Y_2^s$ and for $\xi \geq 0$,

$$F_i^s(a_j, \theta, \xi) = u_i(Y_i^s) + \mathbb{E}\left[\sum_{\tau=1}^{\infty} \delta^\tau u_i(Y_i^{s_\tau}) \mid s_0 = s\right] + \xi.$$

We call this the ξ-RS problem (it is independent of a). When $\xi = 0$, this is the standard risk-sharing problem. Denote by θ^0 the θ which corresponds to the RS case, i.e., $\theta_{ii}^s = 1$ and for $i \neq j$, $\theta_{ij}^s = 0$, $i = 1, 2$, all $s \in \mathcal{S}$. There exists $\hat{\xi}(\theta) > 0$, $\hat{\xi}(\theta) \to 0$ as $\theta \to \theta^0$, such that for all θ ($\theta_{ij}^s \geq 0$, $i, j = 1, 2$, and $\sum_{i=1}^2 \theta_{ij}^s \leq 1$,

$j = 1, 2)$, all $0 \le a_2 \le a_2^*(s)$, all $s \in \mathcal{S}$,

$$\begin{aligned} D_1^s(a_2;\theta) &= \max_{a_1} u_1\left(\theta_{12}^s f_2(a_2) + \theta_{11}^s f_1^s(a_1) - a_1\right) + \\ &\mathbb{E}\left[\sum_{\tau \ge t+1} \delta^{\tau-t} u_1\left(\theta_{12}^{s_\tau} f_2^{s_\tau}\left(a_2^{NE}(s_\tau)\right) + \max_{a_1}\left(\theta_{11}^{s_\tau} f_1^{s_\tau}(a_1) - a_1\right)\right) \mid s_t = s\right] \\ &\le \mathbb{E}\left[\sum_{\tau \ge t} \delta^{\tau-t} u_1\left(Y_1^{s_\tau} + \theta_{12}^{s_\tau} f_2^{s_\tau}(a_2^*)\right) \mid s\right] \\ &\le \mathbb{E}\left[\sum_{\tau \ge t} \delta^{\tau-t} u_1\left(Y_1^{s_\tau}\right) \mid s\right] + \hat{\xi}(\theta), \end{aligned} \tag{S.13}$$

where the first inequality follows from $a_2^{NE}(s) \le a_2^*(s)$ and the fact that if agent 1 deviates, then he gets at most $\theta_{12}^s f_2^s(a_2^*(s))$ more consumption today than his autarkic income Y_1^s, given that $\max_{a_1}(\theta_{11}^s f_1^s(a_1) - a_1) \le Y_1^s$ and $\theta_{12}^s f_2^s(a_2) \le \theta_{12}^s f_2^s(a_2^*)$; likewise in the future given that $\theta_{12}^{s_\tau} f_2^{s_\tau}(a_2^{NE}) \le \theta_{12}^{s_\tau} f_2^{s_\tau}(a_2^*)$. (That is, for any θ near enough to θ^0, we can find a ξ also small such that adding it to autarkic utility in the risk-sharing problem gives a deviation utility bigger than the deviation utility in the hold-up problem.) Likewise for agent 2. Define $[\underline{V}_1^{RS}, \bar{V}_1^{RS}]$ to be the projection of the Pareto frontier onto agent 1's utilities in the RS case in some state s (dropping the dependence on s for notational simplicity, and where possible in what follows). By Ligon et al. (2002), $u_2'(x_2^{RS}(\underline{V}_1^{RS}))/u_1'(x_1^{RS}(\underline{V}_1^{RS})) = \underline{\rho}^{RS}$, $u_2'(x_2^{RS}(\bar{V}_1^{RS}))/u_1'(x_1^{RS}(\bar{V}_1^{RS})) = \bar{\rho}^{RS}$. If autarky can be improved upon, $\underline{V}_1^{RS} < \bar{V}_1^{RS}$ (see main text), and there exists a continuous function $\underline{\xi}(V_1)\colon (\underline{V}_1^{RS}, \bar{V}_1^{RS}) \to \mathbb{R}_{++}$, such that for $\xi \le \underline{\xi}(V_1)$, the solution to the ξ-RS problem exists at $V_1 \in (\underline{V}_1^{RS}, \bar{V}_1^{RS})$ (adapting the arguments in Thomas and Worrall 1988).

A. We prove part (ii) first. Assume that autarky can be improved on in the RS case. Then $[\underline{V}_1^{RS}, \bar{V}_1^{RS}]$ is non-degenerate. Fix $V_1 \in (\underline{V}_1^{RS}, \bar{V}_1^{RS})$. Then, for θ close enough to θ^0 such that $\hat{\xi}(\theta) \le \underline{\xi}(V_1)$, $\{(x(s^t) = w(s^t), a(s^t))\}_{t\ge 0}$ is feasible in the HU problem, where $\{w(t)\}_{t\ge 0}$ solves the $\hat{\xi}(\theta)$-RS problem at V_1 and $a(t) = a^*(s_t)$, which implies (see (S.12b)) at all s^t

$$\begin{aligned} u_1(w_1(s^t)) + \mathbb{E}\left[\sum\nolimits_{\tau \ge t+1} \delta^{\tau-t} u_1(w_1(s^\tau)) \mid s^t\right] &\ge \mathbb{E}\left[\sum\nolimits_{\tau \ge t} \delta^{\tau-t} u_1(Y_1^{s_\tau}) \mid s_t\right] + \hat{\xi}(\theta) \\ &\ge D_1^{s_t}(a_2(s_t);\theta), \end{aligned}$$

where the second inequality follows from (S.13); and also (S.12c) holds trivially. Likewise for agent 2. Thus, the constraints in the HU problem are satisfied. Denote by $C(\theta)$ a solution to the HU problem at V_1 (when it exists) and by $\widetilde{V}_2(C(\theta))$ the corresponding payoff to agent 2, and likewise by $R(\xi)$ and $\widetilde{V}_2(R(\xi))$ the corresponding contract and values in the ξ-RS problem, with $R(0)$ the optimal risk-sharing contract at V_1. We have just shown that for θ close enough to θ^0:

$$\widetilde{V}_2(C(\theta)) \ge \widetilde{V}_2\left(R\left(\hat{\xi}(\theta)\right)\right). \tag{S.14}$$

Let $\theta \to \theta^0$. We assert that $\lim C(\theta) = \lim R(\hat{\xi}(\theta))$ $(= R(0))$. Suppose otherwise, then we can find a subsequence (recall from Lemma 4 that the space of contracts is compact in the usual product topology, and payoffs are continuous in this topology) for which $\lim C(\theta)$ exists and $\lim C(\theta) \ne R(0)$. For this subsequence, then, $\widetilde{V}_2(\lim C(\theta)) = \lim \widetilde{V}_2(C(\theta)) \ge \lim \widetilde{V}_2(R(\hat{\xi}(\theta))) = \widetilde{V}_2(\lim R(0))$ (from (S.14)), but since $\lim C(\theta)$ satisfies the RS constraints ($\hat{z}^s(a^t)$ is maximal in the RS problem, so (S.12c) must hold) and offers agent 2 a payoff at least that in the RS problem, this contradicts the uniqueness of the RS solution. Moreover, for all θ such that $\hat{\xi}(\theta) < \underline{\xi}(V_1)$, neither self-enforcing constraint binds and $a(0) = a^*$. To see this, recall that for $\xi \le \underline{\xi}(V_1)$, the solution to the ξ-RS problem exists at V_1, so that $V_1 \ge \mathbb{E}[\sum_{t\ge 0} \delta^t u_1(Y_1^{s_t}) \mid s_0 = s] + \underline{\xi}(V_1) >$

$\mathbb{E}[\sum_{t\geq 0}\delta^t u_1(Y_1^{s_t}) \mid s_0 = s] + \hat{\xi}(\theta) \geq D_1^{s_0}(a_2;\theta)$ (the latter follows from (S.13)). Hence, agent 1's self-enforcing constraint does not bind, and this also holds for agent 2 because $\widetilde{V}_2(C(\theta)) \geq \widetilde{V}_2(R(\hat{\xi}(\theta))) > \mathbb{E}[\sum_{t\geq 0}\delta^t u_2(Y_2^{s_t}) \mid s_0] + \hat{\xi}(\theta) \geq D_2^{s_0}(a_1;\theta)$. Since neither self-enforcing constraint binds, $a(0) = a^*$. Consequently, with $\mu_1 = \mu_2 = 0$, from (3.2a) with $\mu_i = \nu_i = 0$ $\sigma_r = \sigma_s$ for all such θ, while, (from $\lim C(\theta) = R(0), \sigma_r = \sigma_s$ and (3.2b) with $\nu_i = \gamma_i = 0$, $\sigma_s(V_1) \to u_2'(x_2^{RS}(V_1))/u_1'(x_1^{RS}(V_1))$ because $\theta \to \theta^0$, where x_i^{RS} is agent i's allocation at V_1 in the optimal risk-sharing contract. So far V_1 has been held fixed, but we extend the above arguments for a range of values for V_1. For $\varepsilon > 0$ small enough that $\underline{V}_1^{RS} + \varepsilon < \bar{V}_1^{RS} - \varepsilon$, consider $[\underline{V}_1^{RS} + \varepsilon, \bar{V}_1^{RS} - \varepsilon]$. Since $\underline{\xi}(V_1) > 0$ and continuous on $[\underline{V}_1^{RS} + \varepsilon, \bar{V}_1^{RS} - \varepsilon]$, we can define $\underline{\xi}(\varepsilon) := \min_{V_1 \in [\underline{V}_1 + \varepsilon, \bar{V}_1 - \varepsilon]} \underline{\xi}(V_1)$, where $\underline{\xi}(\varepsilon) > 0$. Thus, for θ such that $\hat{\xi}(\theta) < \underline{\xi}(\varepsilon)$, current actions are efficient (see above) for all $V_1 \in [\underline{V}_1^{RS} + \varepsilon, \bar{V}_1^{RS} - \varepsilon]$, so that $\sigma^r = \sigma^s$ on this interval. Moreover, at $\underline{V}_1^{RS} + \varepsilon$, $\sigma_s(\underline{V}_1 + \varepsilon) \to u_2'(x_2^{RS}(\underline{V}_1^{RS} + \varepsilon))/u_1'(x_1^{RS}(\underline{V}_1^{RS} + \varepsilon))$, and at $\bar{V}_1^{RS} - \varepsilon$, $\sigma_s(\bar{V}_1^{RS} - \varepsilon) \to u_2'(x_2^{RS}(\bar{V}_1^{RS} - \varepsilon))/u_1'(x_1^{RS}(\bar{V}_1^{RS} - \varepsilon))$. Also, for $V_1 \in [\underline{V}_1^{RS} + \varepsilon, \bar{V}_1^{RS} - \varepsilon]$, $\sigma_s(\underline{V}_1^{RS} + \varepsilon) \leq \sigma_s(V_1) \leq \sigma_s(\bar{V}_1^{RS} - \varepsilon)$ by the concavity of the value function. It follows that for any $\eta > 0$ and for all θ close enough to θ^0, $\sigma_r = \sigma_s$ for any $\sigma_s \in [u_2'(x_2^{RS}(\underline{V}_1^{RS} + \varepsilon))/u_1'(x_1^{RS}(\underline{V}_1^{RS} + \varepsilon)) + \eta, u_2'(x_2^{RS}(\bar{V}_1^{RS} - \varepsilon))/u_1'(x_1^{RS}(\bar{V}_1^{RS} - \varepsilon)) - \eta]$. Since ε and η can be made arbitrarily small, and $u_2'(x_2^{RS}(\underline{V}_1^{RS}))/u_1'(x_1^{RS}(\underline{V}_1^{RS})) = \underline{\rho}_s^{RS}$, $u_2'(x_2^{RS}(\bar{V}_1^{RS}))/u_1'(x_1^{RS}(\bar{V}_1^{RS})) = \bar{\rho}_s^{RS}$, the claim in part (ii) of the proposition follows.

B. Maintain the assumption that autarky can be improved upon in RS model. Define $\underline{V}_1(\theta)$ to be the minimum efficient value for V_1 in state s in the HU problem (i.e., where $V_2'(\underline{V}_1(\theta)) = 0$). Recall that agent 1's self-enforcing constraint binds at this point, $D_1(a_2;\theta) = \underline{V}_1(\theta)$ (see Lemma 9). Since $\theta \to \theta^0$, $\underline{V}_1(\theta) = D_1(a_2;\theta) \to \mathbb{E}[\sum_{\tau\geq t}\delta^{\tau-t}u_1(Y_1^{s_\tau}) \mid s_0] = \underline{V}_1$, and consider the sequence of optimal contracts at $\underline{V}_1(\theta)$, denoted by $\underline{C}(\theta)$, as $\theta \to \theta^0$. From the foregoing, $\lim \underline{C}(\theta)$ (as before, taking a convergent subsequence if necessary) yields agent 1 a payoff of $\underline{V}_1$. Let $\underline{R}$ denote the optimal risk-sharing contract at $\underline{V}_1$. We assert that $\lim \underline{C}(\theta) = \underline{R}$. If $\widetilde{V}_2(\lim \underline{C}(\theta)) > \widetilde{V}_2(\underline{R})$, then since $\lim \underline{C}(\theta)$ satisfies the risk-sharing constraints at $\underline{V}_1$ ((S.12c) holds because $z^s(a^{t'})$ is maximal in the RS case), this contradicts the optimality of $\underline{R}$. If $\widetilde{V}_2(\lim \underline{C}(\theta)) < \widetilde{V}_2(\underline{R})$: this implies that we can fix $V_1 > \underline{V}_1$ close enough to $\underline{V}_1$ such that the RS payoff, say, $V_2^{RS}(V_1) > \widetilde{V}_2(\lim \underline{C}(\theta)) + \eta$, for some $\eta > 0$, and from part A, $V_2(V_1;\theta)$ (where we make the dependence of V_2 on θ in the HU problem explicit) is defined for θ close enough to θ^0 and converges to $V_2^{RS}(V_1)$ as $\theta \to \theta^0$. Thus, for θ close enough to θ^0, $|V_2(V_1;\theta) - V_2^{RS}(V_1)| < \eta/2$. Since for θ close enough to θ^0 that $V_1 > \underline{V}_1(\theta)$, $\widetilde{V}_2(\underline{C}(\theta)) = V_2(\underline{V}_1(\theta);\theta) \geq V_2(V_1;\theta)$ by $V_2(\cdot;\theta)$ decreasing, then for all θ close enough to θ^0 we have $\widetilde{V}_2(\underline{C}(\theta)) \geq V_2(V_1;\theta) > \widetilde{V}_2(\lim \underline{C}(\theta)) + \eta/2 = \lim \widetilde{V}_2(\underline{C}(\theta)) + \eta/2$, a contradiction. Thus, $\widetilde{V}_2(\lim \underline{C}(\theta)) = \widetilde{V}_2(\underline{R})$, and so $\lim \underline{C}(\theta) = \underline{R}$, otherwise, there would be two optimal RS contracts at $\underline{V}_1$, which is impossible. Next, at $\underline{V}_1(\theta)$, $\sigma_s = 0$, and $\lim \underline{C}(\theta) = \underline{R}$ implies that $\sigma_r \to u_2'(x_2^{RS}(\underline{V}_1))/u_1'(x_1^{RS}(\underline{V}_1))$. A symmetric argument applies at $\bar{V}_1(\theta)$ defined as the maximum value for V_1 in state s. Given the updating equation is continuous and non-decreasing, and from part A, the claim of part (i) of the proposition then follows (when autarky can be improved upon).

C. Finally, assume no optimal contract improves on autarky in the RS case: So $\underline{V}_1^{RS} = \bar{V}_1^{RS} =: V_1^{AUT}$ say, and there is a unique feasible contract in the RS case, autarky, which we denote R^{AUT}. As in part B, consider the sequence of optimal contracts at $\underline{V}_1(\theta)$, denoted by $\underline{C}(\theta)$, as $\theta \to \theta^0$. $\underline{V}_1(\theta) = D_1(a_2;\theta) \to \mathbb{E}[\sum_{\tau\geq t}\delta^{\tau-t}u_1(Y_1^{s_\tau}) \mid s_0] = V_1^{AUT}$, and likewise $\widetilde{V}_2(\underline{C}(\theta)) \geq D_2(a_1;\theta) \to \mathbb{E}[\sum_{\tau\geq t}\delta^{\tau-t}u_2(Y_2^{s_\tau}) \mid s_0] = \widetilde{V}_2(R^{AUT})$. Hence, $\lim \underline{C}(\theta)$ (as before, taking a convergent subsequence if necessary) yields agent 1 a payoff of V_1^{AUT} and agent 2 a payoff of at least $\widetilde{V}_2(R^{AUT})$. If $\widetilde{V}_2(\lim \underline{C}(\theta)) > \widetilde{V}_2(R^{AUT})$, then since $\lim \underline{C}(\theta)$ satisfies the risk-sharing constraints at V_1^{AUT} ((S.12c) holds because $z^s(a(\tau))$ is maximal in the RS case), this contradicts the optimality of R^{AUT}. Hence, $\widetilde{V}_2(\lim \underline{C}(\theta)) = \widetilde{V}_2(R^{AUT})$ and so $\lim \underline{C}(\theta) = R^{AUT}$ by the uniqueness of the optimal RS contract. Since a symmetric argument applies at $\bar{V}_1(\theta)$, both at $\sigma_s = 0$ and $\sigma_s = \infty$, $\sigma_r \to u_2'(x_2^{RS}(V_1^{AUT}))/u_1'(x_1^{RS}(V_1^{AUT})) = \underline{\rho}_s^{RS} = \bar{\rho}_s^{RS}$, so part (i) of the proposition follows. ∎

Proofs of lemmas for Section 5

For this subsection we maintain Assumptions 1-3 and A.5 and additionally assume that agents are risk-neutral with $u_i(x_i) = x_i$.

Lemma 10. For each $s \in \mathcal{S}$, the Pareto-frontier $V_2^s(\cdot)$ is strictly concave on $[\underline{V}_1^s, \underline{V}_1^{s*})$ where $\underline{V}_1^{s*} := \inf\{V_1: V_2^{s\prime}(V_1) = -1\}$, and on $(\bar{V}_1^{s*}, \bar{V}_1^s]$ where $\bar{V}_1^{s*} := \sup\{V_1: V_2^{s\prime}(V_1) = -1\}$. If first-best actions are not sustainable in state s, i.e, for $s \in \mathcal{S}_c^*$, then $V_2^s(\cdot)$ is strictly concave on $[\underline{V}_1^s, \bar{V}_1^s]$.

Proof. It follows from Proposition 2 that the Pareto-frontier is strictly concave provided that for any two values V_1 and $\widehat{V}_1$, $V_1 \neq \widehat{V}_1$, the corresponding choices satisfy $d \neq \widehat{d}$. If $d(V_1) \neq d^*$, then one or other of the self-enforcing constraints is binding, Therefore, taking a neighborhood about V_1 shows that d cannot be constant on this neighborhood, and hence, that the frontier is strictly concave (on the neighborhood). If, however, $d(V_1) = d^*$, then the actions are first-best, implying $\mu_i = \gamma_i = 0$. Thus, from the first-order condition (3.2b) with $u_i' = 1$, it follows that $V_2'(V_1) = -1$. By concavity, the set of values of V_1 where $V_2'(V_1) = -1$ is an interval (possibly degenerate). Since $V_2^{s(+)}(\underline{V}_1) = 0$ and $V_2^{s(-)}(\bar{V}_1) = -\infty$, this interval is contained in the interior of $[\underline{V}_1, \bar{V}_1]$. ∎

Lemma 11. With probability one, there is a random time $\hat{t} < \infty$ such that $\zeta(t)$ converges monotonically to 0 with $\zeta(t) = 0$ for all $t \geq \hat{t} - 1$.

Proof. Recall the three subsets of $\Lambda^s = [\underline{V}_1^s, \bar{V}_1^s] \subset \mathbb{R}_{++}$: $A^s = \{V_1 \in \Lambda^s: c_1^o = 0\}$, $B^s = \{V_1 \in \Lambda^s: c_1^o > 0 \text{ and } c_2^o > 0\}$ and $C^s = \{V_1 \in \Lambda^s: c_2^o = 0\}$ where (c_1^o, c_2^o) represents an optimal value for consumption at V_1. For notational convenience, we drop the state superscripts and define $\sigma(t{+}1) := \sigma^+(t)$. Recall that $\zeta(t) = \max\{\zeta_1(t), \zeta_2(t)\}$ and that from the first-order conditions $\zeta_1(t) = -\ln(\sigma(t{+}1))$ and $\zeta_2(t) = \ln(\sigma(t{+}1))$. First, if $\sigma(t) = 1$, then $\sigma(t{+}1) = 1$. Using (3.2b), this is immediate if $V_1 \in B$. It also follows that $\sigma(t{+}1) = 1$ for $V_1 \in A$ or $V_1 \in C$ because, for $V_1 \in A$, $1 \geq \sigma^+ \geq \sigma$ and for $V_1 \in C$, $1 \leq \sigma^+ \leq \sigma$. Next, suppose w.l.o.g. that $\sigma_0 < 1$. Since $\sigma \geq 1$ for $V_1 \in C$, it follows that $V_1 \in A$ or $V_1 \in B$. For $V_1 \in B$, $\sigma(1) = 1$, and for $V_1 \in A$, $1 \geq \sigma^+ \geq \sigma$. Hence, $1 \geq \sigma(t{+}1) \geq \sigma(t)$. Thus, $\zeta(t)$ declines for all σ_0. It remains to establish that convergence to $\sigma(t) = 1$ occurs. Let t' be the random period when $c_1 > 0$ first occurs. We first show that $t' < \infty$ almost surely. Notice that by virtue of $a_1 \geq a_1^{NE}(s) > 0$ for any state s, when $c_1 = 0$, agent 1's utility is at most $-a_1^{NE}(s)$. Let $-\underline{a}_1 := \max_{s \in \mathcal{S}}\{-a_1^{NE}(s)\} < 0$. Since net utilities are bounded in equilibrium, denote by $\bar{u}_1$ the maximum utility to agent 1 in any state. Let τ be such that $\delta^\tau \bar{u}_1/(1-\delta) < \underline{a}_1$. Then, starting in any state s at any date t, it must be the case that $c_1 > 0$ on some positive probability path within the next τ periods because otherwise future utility after $t+\tau$ cannot compensate the current negative utility. Letting $\underline{\pi}$ be the minimum probability of any such τ-period path (that is, the minimum probability of a positive probability path), we conclude that after history s^t, there is a probability of at least $\underline{\pi} > 0$ that $c_1 > 0$ before $t + \tau$. Consequently, $\Pr[\,\exists\, t \text{ such that } c_1(t) > 0\,] = 1$. From the above argument, we have $\sigma(t) \leq 1$, but if $c_1(t) > 0$ at t, then $V_1 \in B$ or $V_1 \in C$. If $V_1 \in B$, then $\sigma(t{+}1) = 1$; if $V_1 \in C$, then $\sigma(t) \geq 1$, and hence, combining inequalities, $\sigma(t) = 1$. Hence, $\Pr[\,\exists\, t \text{ such that } \zeta(t) = 0\,] = 1$. ∎

Lemma 12. For each $s \in \mathcal{S}$, the surplus function $z^s(V_1)$ is a continuous and single peaked function of V_1. That is, for any $V_1^{(1)} < V_1^{(2)} < V_1^{(3)}$, it is not possible that $z^s(V_1^{(1)}), z^s(V_1^{(3)}) > z^s(V_1^{(2)})$. Moreover, $z^s(V_1)$ is maximal when $\sigma_s(V_1) = 1$.

Proof. Continuity follows from Lemma 5. Suppose, to the contrary, that $z^s(V_1)$ is not single peaked and that there is $V_1^{(1)} < V_1^{(2)} < V_1^{(3)}$ such that $z^s(V_1^{(1)}) > z^s(V_1^{(2)})$ and $z^s(V_1^{(3)}) > z^s(V_1^{(2)})$. By concavity of the Pareto-frontier, there is some $\lambda \in (0,1)$ such that the convex combination of contracts satisfies $V_1^{(\lambda)} = \lambda V_1^{(1)} + (1-\lambda)V_1^{(3)} \leq V_1^{(2)}$ and $V_2^{(\lambda)} = \lambda V_2^s(V_1^{(1)}) + (1-\lambda)V_2^s(V_1^{(3)}) \leq V_2^s(V_1^{(2)})$. Let $d_i^{(k)}$ and $c_i^{(k)}$ denote the optimal choices at $V_1^{(k)}$ for $k = 1,2,3$. In addition, let $d_i^{(\lambda)} = \lambda d_i^{(1)} + (1-\lambda)d_i^{(3)}$. in the proof of Proposition 2 it is shown that under Assumption A.5 surplus is a concave function of d. Hence, $z^s(d_1^{(\lambda)}, d_2^{(\lambda)}) \geq \lambda z^s(d_1^{(1)}, d_2^{(1)}) + (1-\lambda) z^s(d_1^{(3)}, d_2^{(3)}) = \lambda z^s(V_1^{(1)}) + (1-\lambda) z^s(V_1^{(3)}) \geq \min(z^s(V_1^{(1)}), z^s(V_1^{(3)})) > z^s(V_1^{(2)})$. Now consider the contract at $V_1^{(2)}$ and replace $d_i^{(2)}$ by $d_i^{(\lambda)}$, and replace $c_i^{(2)}$ by $\tilde{c}_i$, such that $\tilde{c}_i -$

$g_j^s(d_i^{(\lambda)}) > c_i^{(2)} - g_j^s(d_i^{(2)})$ and $\tilde{c}_1 + \tilde{c}_2 - g_2^s(d_1^{(\lambda)}) - g_1^s(d_2^{(\lambda)}) = z^s(d_1^{(\lambda)}, d_2^{(\lambda)})$. The existence of such $\tilde{c}_i$ is guaranteed by the strict inequality just established that $z^s(d_1^{(\lambda)}, d_2^{(\lambda)}) > z^s(d_1^{(2)}, d_2^{(2)})$. While making these changes to current utilities, keep continuation utilities unchanged. The new utilities satisfy $V_i > V_i^{(2)} \geq V_i^{(\lambda)} \geq d_j^{(\lambda)}$, where the first inequality follows from the construction of the new contract, the second inequality follows from the concavity of the frontier and the choice of λ, and the third follows from the definitions of $V_i^{(\lambda)}$ and $d_J^{(\lambda)}$ and the constraints $V_i^{(k)} \geq d_j^{(k)}$. This provides a contradiction, and hence, we conclude that $z(V_1)$ is single peaked.

Consider $\sigma_s(V_1) = 1$. If $V_1 \in A^s$ or $V_1 \in C^s$, then it follows from the first-order conditions that actions and surplus are first-best. For $V_1 \in B^s$, there are two possibilities: either $\mu_1 = \mu_2 = 0$, or $\mu_1, \mu_2 > 0$. In the former case, actions and surplus are first-best. In the latter case, note that the first-order conditions can be used to show $\mu_1 = (\partial z^s/\partial d_2)/(1 - (\partial z^s/\partial d_1))$ and $\mu_2 = (\partial z^s/\partial d_1)/(1 - (\partial z^s/\partial d_1))$. It has already been shown in the proof of Proposition 2 that $(1 - (\partial z^s/\partial d_1)) \neq 0$ except possibly where $V_1 = \bar{V}_1^s$. Hence, from Lemma 5 and the continuity of the functions z^s (and g_i^s), the multipliers are continuous *functions* of V_1 and $\mu_1, \mu_2 > 0$ in an open neighborhood of V_1. Thus, in this neighborhood, $d_2(V_1) = V_1$ and $d_1(V_1) = V_2^s(V_1)$. Since $V_2(\cdot)$ is a differentiable function, d_i is a differentiable function of V_1 in this neighborhood, with derivatives $\mathrm{d}d_2/\mathrm{d}V_1 = 1$ and $\mathrm{d}d_1/\mathrm{d}V_1 = -\sigma_s(V_1)$. Hence,

$$\frac{\mathrm{d}z^s(V_1)}{\mathrm{d}V_1} = -\sigma_s(V_1)\frac{\partial z^s(d_1, d_2)}{\partial d_1} + \frac{\partial z^s(d_1, d_2)}{\partial d_2}.$$

It can also be checked from the first-order conditions that the derivative $\mathrm{d}z^s(V_1)/\mathrm{d}V_1$ is zero when $\sigma_s(V_1) = 1$. Moreover, it can be seen that $z^s(V_1)$ is concave in this neighborhood, and hence, the surplus is maximal. ■

Supplementary Material References

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