Time Consistency and Intergenerational Risk Sharing

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Abstract

It is shown how intergenerational risk sharing can be achieved by transfers from the young generation to the old generation such that the young generation will never have an incentive to unilaterally renege on the transfer. This contradicts a claim made in Gordon and Varian (1988) that intergenerational risk-sharing is infeasible because of problems of time consistency. It is shown however, that even in a stationary environment, time consistent transfers are non-stationary when utility is monotone increasing.

Keywords: Intergenerational risk-sharing; social contract; time inconsistency; self-enforcing.

JEL Classification: D91; E61; H55.

1. Introduction

Economic efficiency dictates that where possible risk be shared. Any single generation can share idiosyncratic but not aggregate risk. Aggregate or social risk for one generation is however, idiosyncratic when compared with the social risks of different generations. Therefore there is a potential benefit to sharing risk between generations.

A large literature has explored these potential benefits from intergenerational risk sharing using a stochastic overlapping generations framework. This literature has considered ways in which both *interim* and *ex ante* efficiency can

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be enhanced through market mechanisms and pay-as-you-go social security schemes under a variety of different assumptions about production possibilities and the stochastic processes generating risk.[1] In addition a number of papers, such as Krueger and Kubler (2006) and Gottardi and Kubler (2006), have used models of this type to estimate the welfare gains from social security reform by improving risk sharing across generations.

Despite the success of this literature less attention has been paid to the issue of the time consistency of any social security scheme. Whilst it might be possible to devise risk sharing programs such that every generation benefits from the scheme at the time of birth, *ex post* there may be states where agents will not voluntarily contribute to the system. If such states occur then the scheme is not time consistent. One reason why less attention has been paid to the time consistency problem is that in their seminal article Gordon and Varian (1988) came to the stark conclusion that "*an intergenerational risk-sharing scheme is infeasible due to problems of time consistency*".[2] Thus they conclude that without some commitment intergenerational risk sharing is impossible.

The impossibility of time consistent intergenerational risk sharing however, sits somewhat uneasily with the strategic "pension games" considered by Hammond (1975) and the folk theorem conclusions of non-stochastic overlapping generation games, e.g. Crémer (1986), Kandori (1992) and Smith (1992). Consider, for example, the case where the old and young play a prisoners' dilemma game. The old will always defect as there is no future payoff. Nevertheless the young may play co-operate in the expectation that the future young will also play co-operate against them when they are old. If the young deviate to defect, then subsequent play is for all future players to defect at each period, which is the one-shot Nash equilibrium. If the discount factor is large enough, the threat is credible and the players will prefer to co-operate when young. In the limit if the discount factor is close enough to one, then a folk theorem result emerges that

[1] See e.g. Weiss (1980), Enders and Lapan (1982), Smith (1982), Gordon and Varian (1988), Hassler and Lindbeck (1997), Thøgersen (1998), Marini and Scaramozzino (1999), Demanage (2002), Thøgersen (2003) and Demanage (2005).

[2] Gordon and Varian (1988), p.195.

any individually rational payoffs are sustainable as a sub-game perfect equilibrium. With this interpretation social security might arise from a social contract between the generations and is enforced by a threat of breakdown in cooperation without a need for any external commitment devise.

This paper shows that these two strands of the literature are not completely inconsistent. Intergenerational risk sharing is possible through a purely social contract between generations. Section 4 shows that Gordon and Varian (1988) were correct in asserting that there can be no time consistent intergenerational risk sharing scheme if utility is monotone increasing and transfers are restricted to be *stationary*. However, the pension games approach suggests that by relaxing the assumption of stationarity and allowing the risk sharing scheme to exhibit *history dependence* it will be possible to share risk across generations and achieve time consistency. Section 5 confirms this supposition. Since the purpose here is only to show existence of time consistent intergenerational risk sharing Section 5 constructs an example using a simple two-state variant of the Gordon-Varian model. Moreover, it shows that it is possible to construct time consistent intergenerational risk sharing schemes that are actuarially fair. In these actuarially fair schemes the expected transfer of the young have a martingale property so that on average each young generation pays into the scheme the same amount as the immediately proceeding generation.

The paper proceeds as follows. Section 2 presents the Gordon-Varian model. Section 3 considers feasible transfer schemes when commitment exists. Section 4 considers the time consistency issue and Section 5 constructs examples of time consistent risk sharing schemes. Section 6 discusses these results and possible directions for future research.

2. The Simple Gordon-Varian Model

Gordon and Varian (1988) provided a simple and powerful overlapping generations model of intergenerational risk sharing which has been used extensively in subsequent work such as Thøgersen (1998) and Marini and Scaramozzino (1999). Here we present the simplest version of the Gordon-Varian model. Agents live for two periods. They earn a non-random wage, w when young and consume only when old. If the young agent has an income of w, then it is saved and produces a return of $w - \varepsilon$,[3] where ε is a random variable with mean $\mathrm{E}[\varepsilon] = 0$ and variance $\mathrm{Var}[\varepsilon] = \mathrm{E}[\varepsilon^2] = \sigma^2 > 0$. The realizations of ε is the same for all members of the generation and occurs after the previous generation has died but before the new generation is born, so that all risk in the economy at any one point in time is aggregate risk. For simplicity there is no population growth. It is assumed that utility when old depends on the mean and variance of consumption according to a mean-variance utility function $U = \mathrm{E}[c] - \mathrm{Var}[c]$,[4] where $c = w - \varepsilon$ is the random old age consumption. The ε are *i.i.d.* random variables, so that absent any transfers, each generation has an autarky utility of $w - \sigma^2$.

This model is well-suited to its purpose of examining intergenerational risk sharing. To see this suppose that $\varepsilon \equiv 0$ so that there is no uncertainty. It has been shown by Samuelson (1958) and Shell (1971) in non-stochastic overlapping generations models that a Pareto-improvement can sometimes be found where each generation passes resources backward to the previous generation. This is also the case here. To see this let x_t denote the transfer from the young alive at time t to the old alive at time t and let $x_t = 1 - \beta^t$ for $\beta \in (0,1)$. Then consumption of the old at date t is $c_t = w - x_{t-1} + x_t = w + (1-\beta)\beta^{t-1} > w$. Thus each generation receives a slightly greater transfer when old than it makes when young and hence consumption is raised above w but to a diminishing extent for each generation. Such schemes are not actuarially fair in that each generation transfers more into the scheme than the previous generation as $x_t > x_{t-1}$.[5] . When examining intergenerational risk-sharing we shall therefore want to consider only transfer schemes that do not require transfers to increase on average from one

[3] This is the simplest model presented in Gordon and Varian (1988) but the sign on ε reversed so that the transfers to the old defined below are positive when the shock is negative and income reduced ($\varepsilon > 0$).

[4] This is the standard constant absolute risk aversion mean-variance utility function $U = \mathrm{E}[c] - (1/2)A\,\mathrm{Var}[c]$, with $A = 2$. We shall for convenience frequently use the mean-variance framework but will also demonstrate that the key results do not depend on this choice of utility function and apply equally for any strictly increasing and strictly concave utility function.

[5] If $\beta = 0$ then $x_t = 1$ and only the first generation benefits.

generation to the next, so that absent uncertainty the welfare of all generations cannot be improved.

3. WITH COMMITMENT

To examine intergenerational risk sharing we suppose that the transfer x_t from the young alive at time t to the old alive at time t can be conditioned on current and past shocks. With uncertainty we shall want to allow $x_t < 0$ so that there may be a transfer from the old to the young. We first consider as a benchmark the case in which there is full commitment and it is possible to enforce the transfer x_t. The following scheme of transfers works quite well. Let $x_1 = \varepsilon_1$. The consumption of the old at time $t = 1$ is therefore w with a utility of w since the randomness is completely eradicated. The return of the old at time $t = 2$ is $w - x_1 - \varepsilon_2 = w - \varepsilon_1 - \varepsilon_2$, therefore let $x_2 = \varepsilon_1 + \varepsilon_2$ so that again consumption of the old at time $t = 2$ is certain and equal to w with utility of w. In general the transfer will satisfy[6]

$$\text{(GV}^*\text{)} \qquad x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \varepsilon_t = \sum_{i=1}^{t} \varepsilon_i & \text{For } t \geq 1. \end{cases}$$

With this transfer scheme GV* the consumption of the old alive at time t is $c_t = w - x_{t-1} - \varepsilon_t + x_t = w$ which is independent of the the shock ε_t. Essentially since risk is all idiosyncratic across the generations it can be effectively eliminated by pooling over all generations, *all good or bad luck is transferred to the future*. This scheme though has an important feature that makes it undesirable. Although $\mathrm{E}[x_t] = 0$ so on average each generation transfers nothing to the current old generation the variance of the transfer is $\mathrm{Var}[x_t] = t\sigma^2$, so the transfer is growing without bound. As there is no borrowing it is natural that any feasible scheme should satisfy a non-negativity constraint on the income of the young at date t:

$$\text{(1)} \qquad y_t = w - x_t \geq 0 \qquad \forall t.$$

[6]The reason for the label will be explained shortly.

As the variance of the transfer is unbounded, no matter how large is w, this transfer scheme will violate (1) with probability one in finite time.[7]

The scheme actually considered by Gordon and Varian (1988) is a variant of the above scheme that has equal risk-sharing between $n \geq 2$ generations.

$$(\mathrm{GV}^n) \qquad x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{t-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t < n \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{n-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t \geq n. \end{cases}$$

Then consumption of the old alive at time t is $c_t = w - x_{t-1} - \varepsilon_t + x_t = w - \sum_{i=0}^{n-1} \frac{\varepsilon_{t-i}}{n}$ with $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \frac{\sigma^2}{n}$ for $t \geq n$. The utility of the old alive at time t is $u = w - \frac{\sigma^2}{n}$. The scheme works best when n is large and as $n \to \infty$ utility tends to the first-best level. Again however there is a conflict with the inequality (1). The variance of the transfer is given by[8]

$$(2) \qquad \mathrm{Var}[x_t] = \frac{n-1}{n} \frac{2n-1}{6} \sigma^2.$$

This is increasing without bound in n so that as n is increased (1) is violated, and hence the scheme will become infeasible, with probability one in finite time.

Another simple way to bound the variance of the transfer (apart from choosing n small) is to have less risk-sharing over time. A simple scheme that does this is

$$(\mathrm{DRS}) \qquad x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \beta^{t-1} \varepsilon_t & \text{For } t \geq 1 \end{cases}$$

[7]The scheme GV* has a Ponzi-like property that the transfer the young make at date t is refunded by the transfer of young at date $t+1$. However, it is not strictly a Ponzi game (see O'Connell and Zeldes, 1988) as there is no borrowing. Like a Ponzi game, in which the discounted value of debt is strictly positive for all long enough time horizons, the scheme GV* has the property that constraint (1) .

[8]This variance is calculated from the recurrence relation and the proof is given in the Appendix.

where $\beta \in (0,1)$. This offers complete insurance at date $t = 1$ as $x_1 = \varepsilon_1$. Consumption at time t is $c_t = w - (1-\beta^{t-1})\varepsilon_t$, so $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \sigma^2(1 - \beta^{t-1}(2-\beta^{t-1}))$ which is increasing over time. The utility of the generation born at $t-1$ is $w - \sigma^2(1-\beta^{t-1}(2-\beta^{t-1}))$, which although declining over time toward the autarky level, provides a Pareto-improvement over autarky for each generation. Moreover the expected transfer is $\mathrm{E}[x_t] = 0$ and although the variance of the transfer is increasing over time, $\mathrm{Var}[x_t] \to \sigma^2/(1-\beta^2)$ as $t \to \infty$. Thus provided the support of ε is small relative to w the Pareto-improvement can be achieved without violating (1).

4. Time Consistency and Self-enforcement

In this section we consider the sequence of transfers as a kind of social contract in which any generation may freely renege if it is to their advantage. Let $h^t = (\varepsilon_1, \varepsilon_2, \ldots \varepsilon_t)$ be a history of shocks. A social contract is a sequence of transfers $(x_t(h^t))$ such that no generation will find it to their advantage to renege on any transfer $x_t(h^t)$. The social contract[9] then has to satisfy two types of constraint. First it must satisfy an *ex post* or time consistency constraint requiring the old should not lose in making a transfer and secondly an *interim* or self-enforcing constraint that the young can benefit from making the prescribed transfer given the expectation of a future transfer as specified in the social contract.

The time-consistency constraint for the old generation at time t is

$$u(w - \varepsilon_t - x_{t-1}(h^{t-1}) + x_t(h^{t-1}, \varepsilon_t)) \geq u(w - x_{t-1}(h^{t-1}))$$

for all ε_t. It is therefore clear that the old will never make a transfer to the young as they have nothing to gain from such a transfer no matter what the utility function. Thus we must have

$$x_t(h^t) \geq 0 \qquad \forall h^t. \tag{3}$$

[9]The notion of a contract is used here in its loosest sense as it may be purely implicit with nothing written down and with no recourse to legal remedies if the contract is not adhered to.

This is exactly as in the prisoners' dilemma game described in the introduction. The old never have an incentive to co-operate. Nevertheless the young may still have an incentive to make a transfer in anticipation of transfers to them by the future young. This is an important feature of the time consistent social contract, any contract has a contingent pension provision. Thus in the terms of Rangel and Zeckhauser (2001) all trade involves backward transfers with no forward transfers from old to young generations. Notice that neither the scheme GV^* nor the Gordon-Varian n generation scheme GV^n nor the declining risk-sharing scheme DRS discussed in Section 3 respect this constraint since they all involve forward transfers from the old to the young after some histories. None of these schemes is therefore time consistent.

We now consider the self-enforcement constraint faced by the young generation. We've supposed that any failure by the young at time t to make the appropriate transfer results in a breakdown of trust and the social contract dissolves thereafter, $x_\tau = 0 \quad \forall \tau > t$. Then clearly if the young at time t are to renege they will choose $x_t = 0$ and this is a Nash equilibrium exactly as in the prisoners' dilemma game. The young may have an incentive to make the current transfer if they anticipate that the young next period will also make a transfer next period if they suffer a bad shock and $\varepsilon > 0$. Consumption of the old at time $t+1$ is $c_{t+1}(h^{t+1}) = w - x_t(h^t) - \varepsilon_{t+1} + x_{t+1}(h^{t+1})$. For a given value of x_t, the constraint for the young at time t is

$$\text{(4)} \qquad \mathrm{E}[u(w - \varepsilon_{t+1} - x_t(h^t) + x_{t+1}(h^t, \varepsilon_{t+1}))|h^t] \geq \mathrm{E}[u(w - \varepsilon_{t+1})].$$

This equation says that the young at time t, knowing the shock ε_t and the called for transfer in that state, $x_t(h^{t-1}, \varepsilon_t)$, will prefer to make the transfer given what they can expect in their old age, rather than making no transfer and receiving no transfer in their old age in return. The constraint (4) makes it clear that there can be no intergenerational risk sharing with stationary transfers. To see this suppose that the transfer x depends only on the current shock, ε, say that $x = g(\varepsilon)$. Since the constraint must hold for any ε_t, it hold for $x_{max} = \max\{g(\varepsilon_t)\}$. But then for any monotone increasing function $u(w - \varepsilon_t - x_{max} + g(\varepsilon_t)) \leq u(w - \varepsilon_t)$

as $g(\varepsilon_t) - x_{max} \leq 0$. Thus if any intergenerational risk sharing can be achieved, it cannot be achieved through a stationary transfer scheme.

The inequality (4) can be expressed even more simply in the mean-variance framework of Gordon-Varian. The expected consumption at time $t+1$ is $\mathrm{E}[c_{t+1}] = w - x_t + \mathrm{E}[x_{t+1}]$ and the variance of consumption at date $t+1$ is $\mathrm{Var}[c_{t+1}] = \sigma^2 + \mathrm{Var}[x_{t+1}] - 2\,\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}]$. The young at time t will not renege on the transfer x_t provided

$$w - x_t + \mathrm{E}[x_{t+1}] - (\sigma^2 + \mathrm{Var}[x_{t+1}] - 2\,\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}]) \geq w - \sigma^2 \qquad \forall t.$$

This can be rewritten in terms of the transfer at date t as

$$x_t \leq \mathrm{E}[x_{t+1}] - \mathrm{Var}[x_{t+1}] + 2\,\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}] \qquad \forall t. \tag{5}$$

The mean variance formulation is convenient because if the social contract provides future insurance then $\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}] \geq 0$. Thus the inequality (5) demonstrates that the young will make the transfer when they expect the transfer next period to be high enough and when the future insurance offered is sufficiently great.

5. Example

We shall now be interested in finding a risk-sharing arrangement that is feasible, i.e. satisfies the non-negativity requirement (1), is time consistent i.e. satisfies (3) and is self-enforcing i.e. satisfies (4) in general and (5) in the mean-variance context. As stationary schemes of intergenerational risk-sharing cannot be supported in general, the calculation of the optimum non-stationary scheme is difficult. Thus the purpose of this paper is only to show that it is possible to have time consistent intergenerational risk sharing. It is therefore sufficient to construct examples which show that such schemes exist. In this section it is shown that it is possible to have a time consistent social contract of intergenerational risk-sharing. Here we consider a simple two-state example. Suppose $\varepsilon \in \{-\sigma, \sigma\}$ with equal probability and assume that $w > \sigma$. Two schemes will be considered; a stationary scheme which satisfies the constraints for the mean-

		date t	
		$\varepsilon_t = -\sigma$	$\varepsilon_t = \sigma$
date $t-1$	$\varepsilon_{t-1} = -\sigma$	$w+\sigma$	w
	$\varepsilon_{t-1} = \sigma$	w	$w-\sigma$

Table 1: Consumption at date t for two generation scheme

variance utility and a non-stationary scheme which satisfies the constraints for any risk averse utility function.

Before considering these two schemes we first examine the two benchmarks of autarky and the Gordon-Varian two generation scheme. In autarky consumption is either $w-\sigma$ or $w+\sigma$ with equal probability. Hence expected consumption is $\mathrm{E}[c_t] = w$ and the variance of consumption is $\mathrm{Var}[c_t] = \sigma^2$. Expected utility is $w-\sigma^2$. In the Gordon-Varian two generation scheme

$$x_t = x_{t-1} + \varepsilon_t - \frac{1}{2}(\varepsilon_t + \varepsilon_{t-1}); \qquad x_0 = 0;\ \ \varepsilon_0 = 0. \tag{GV^2}$$

So $c_t = w - x_{t-1} - \varepsilon_t + x_t = w - \frac{1}{2}(\varepsilon_t + \varepsilon_{t-1})$. Thus consumption takes on one of four possible values shown in Table 1, each cell occurring with equal probability We have $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \frac{1}{2}\sigma^2$. So that utility is $w - \frac{1}{2}\sigma^2$, for each generation beyond the first (the first generation have utility $w - \frac{1}{4}\sigma^2$). Clearly this scheme does not satisfy the time consistency constraint, (3), e.g. $x_1 = \frac{1}{2}\varepsilon_1$ which is negative if $\varepsilon_1 = -\sigma$. It does however satisfy the other two constraints. To check (5) we take x_{t-1} and ε_{t-1} as given, to find $\mathrm{E}[x_t] = x_{t-1} - \frac{1}{2}\varepsilon_{t-1}$, $\mathrm{Var}[x_t] = \frac{1}{4}\sigma^2$ and $\mathrm{Cov}[x_t, \varepsilon_t] = \frac{1}{2}\sigma^2$, so $\mathrm{E}[x_t] - (\mathrm{Var}[x_t] - 2\,\mathrm{Cov}[x_t, \varepsilon_t]) = x_{t-1} - \frac{1}{2}\varepsilon_{t-1} + \frac{3}{4}\sigma^2$. Thus (5) is satisfied as $\varepsilon_{t-1} \le 1\frac{1}{2}\sigma^2$. With $n = 2$, $x_t = \pm\frac{1}{2}\sigma^2$, so that $\mathrm{E}[x_t] = 0$ and (1) is satisfied.

Since the mean-variance utility is non-monotonic it is actually possible to find a stationary scheme that is time consistent provided σ is not too small.[10] To see this consider the following simple stationary scheme which does just as well

[10] If the risk-sharing scheme is to satisfy the constraints, then the risk-sharing benefits should not be too small.

as the two-generation scheme but satisfies the constraints (1), (3) and (5):

$$\text{(SMV)} \qquad \begin{array}{rclll} x_t & = & \sigma & if & \varepsilon_t = \sigma \\ x_t & = & 0 & if & \varepsilon_t = -\sigma. \end{array}$$

This scheme is stationary because the transfer depends only on the current state and not the past history. It satisfies (3) by construction. Constraint (1) is satisfied as x_t is at most σ and by assumption $w > \sigma$. It also clearly satisfies the requirement (3). With $\mathrm{E}[x_t] = \frac{1}{2}\sigma$, $\mathrm{Var}[x_t] = \frac{1}{4}\sigma^2$ and $\mathrm{Cov}[x_t, \varepsilon_t] = \frac{1}{2}\sigma^2$, $\mathrm{E}[x_t] - (\mathrm{Var}[x_t] - 2\,\mathrm{Cov}[x_t, \varepsilon_t]) = \frac{1}{2}\sigma^2 - (\frac{1}{4} - 1)\sigma^2 = 1\frac{1}{4}\sigma^2$. Since x_{t-1} is at most σ, the constraint (5) is always satisfied. To be explicit, suppose that $\varepsilon_{t-1} = \sigma$ and the young at time $t-1$ are required to make a transfer of σ to the old. If they do so, and the contract is adhered to by future generations, then their consumption next period will be either $w - \sigma$ if $\varepsilon_t = \sigma$ or w if $\varepsilon_t = -\sigma$. Expected consumption is $w - \frac{1}{2}\sigma$ and the variance of consumption is $\frac{1}{4}\sigma^2$. Thus the utility at date t conditional on transferring σ at date $t-1$ is $w - \frac{1}{2}\sigma - \frac{1}{4}\sigma^2$. This is to be compared with the autarky utility of not making the transfer and receiving no transfer at date t which yields a utility payoff of $w - \sigma^2$. Thus the transfer scheme SMV is an equilibrium if $\sigma > 2/3$. Since x_t does not depend on x_{t-1} clearly the self-enforcing constraint (5) is also satisfied when no transfer is required at date $t-1$.

It is to be remembered that viewed ex ante x_{t-1} is also a random variable. So consumption $c_t = w - x_{t-1} - \varepsilon_t + x_t$ is the same as the two generation scheme given in Table 1. As each cell occurs with equal probability, $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \frac{1}{2}\sigma^2$. Ex ante utility is $w - \frac{1}{2}\sigma^2$, for each generation. This is an improvement over autarky. The improvement however, hinges on the mean-variance framework. Conditional on $\varepsilon_{t-1} = \sigma$ consumption of the old under autarky in period t is $w - \sigma$ if $\varepsilon_t = \sigma$ and $w + \sigma$ if $\varepsilon_t = -\sigma$. With the transfer scheme in SMV, conditional on $\varepsilon_{t-1} = \sigma$ and the transfer being made in period $t-1$, consumption of the old in period t is $w - \sigma$ if $\varepsilon_t = \sigma$ and w if $\varepsilon_t = -\sigma$. Thus autarky stochastically dominates the scheme SMV conditional on $x_{t-1} = 1$.

Next we consider a non-stationary scheme that does provide inter-generational risk-sharing for any risk-averse utility function. By the previous result such a scheme must be non-stationary. However, we shall impose an additional constraint that the scheme is actuarially fair. Actuarially fair schemes will satisfy the martingale property

$$\mathrm{E}[x_t] = x_{t-1}$$

for almost all histories h^t so that each generations expects to pay into the scheme exactly the same on average as the proceeding generation. Starting from $x_0 = 0$, let

$$x_t = \max[0, x_{t-1} + \beta^{t-1}\varepsilon_t] \tag{AF}$$

where $\beta \in (0,1)$. In scheme AF if $x_t > 0$, we have $c_t = w \pm (1-\beta^{t-1})\sigma$. Transfers satisfy the martingale property that $\mathrm{E}[x_{t+1} \mid x_t > 0] = x_t$ and $\mathrm{E}[c_{t+1} \mid x_t > 0] = w$. Further the variance of consumption is

$$\mathrm{Var}[c_{t+1} \mid x_t > 0] = \left(1 - \beta^t(2-\beta^t)\right)\sigma^2$$

which is less than the autarky variance of σ^2. Thus for $x_t > 0$, the scheme has a mean preserving reduction in risk and for any risk averse utility function will be preferred to autarky. The only histories where $x_t = 0$ are those where there has been no positive shock in the past, i.e. $\varepsilon_\tau = \sigma$ for all $\tau < t$. When $x_t = 0$ the scheme provides a no less consumption in each state. Consumption is undiminished at $w + \sigma$ when $\varepsilon = -\sigma$ and is $w - (1-\beta^{t-1})\sigma > w - \sigma$ when $\varepsilon = \sigma$. Thus when $x_t = 0$ the scheme is clearly preferred over autarky. In this case $\mathrm{E}[x_{t+1} \mid x_t = 0] = \sigma/2$. The effect of this is that although $\mathrm{E}[x_t] = x_{t-1}$ along almost all histories, there is still growth in x_t. This has to be the case for any risk sharing scheme which satisfies (3) starting from $x_0 = 0$.

6. Conclusion

It has been shown that it is possible to have some intergenerational risk sharing even when each generation can unilaterally renege on any risk sharing transfers if it is to their own advantage. This contradicts the claim made in Gordon and Varian (1988) that in these circumstances no intergenerational risk sharing is possible. Further it has been shown that any time consistent intergenerational risk sharing must take the form of a non-stationary state contingent pension payable by the young to the old generation if the utility function is monotonically increasing.

Although the Gordon-Varian model is very useful for considering inter-generational risk-sharing and has been the basis for a number of studies there is much that is left out of account. First since utility depends on consumption when old there is no consumption-investment trade-off for the young and all resources are invested. Secondly the only source of uncertainty is the rate of return on savings. In a more general model there could be uncertainty to the endowments of the young and old generations. Thirdly there is no physical long-lived asset such as capital or land and all savings depreciate after one period. Fourthly there in no per-capita or population growth. All of these factors would be important extensions to the model. Moreover, these extensions might prove quite critical. Consider for example, the case where the expected returns on savings is strictly positive. The transfers from the old to the young are inefficient because expected return is forgone. Risk-sharing might still be achieved if the benefits from risk-sharing outweigh this inefficiency. However

Future work should be directed at finding optimum time consistent schemes and studying more general stochastic overlapping generations models with growth and capital accumulation.

APPENDIX

Proof of equation (2) that $\text{Var}[x_t] = \frac{n-1}{n}\frac{2n-1}{6}\sigma^2$

The rule for determination of the transfer is:

$$x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{t-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t < n \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{n-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t \geq n. \end{cases}$$

Repeated substitutions to give

$$x_t = \sum_{i=0}^{n-1} \frac{n-1-i}{n} \epsilon_{t-i}$$

or writing it out in full

$$x_t = \frac{n-1}{n}\epsilon_t + \frac{n-2}{n}\epsilon_{t-1} + \frac{n-3}{n}\epsilon_{t-2} + \cdots + \frac{(n-1)-(n-2)}{n}\epsilon_{t-(n-2)} + \frac{(n-1)-(n-1)}{n}\epsilon_{t-(n-1)}.$$

Since the $\text{Var}[\epsilon_t] = \sigma^2$ and $Cov[\epsilon_{t-i}, \epsilon_{t-j}] = 0$ for $i \neq j$, the variance of x_t is determined by the sum of the squares of the coefficients. Thus it is necessary to calculate the partial sum

$$s_n = 1^2 + 2^2 + 3^2 + \cdots + (n-2)^2 + (n-1)^2.$$

One simple way to calculate this sum is to use the method of difference sequences which shows that

$$s_n = 1\binom{n}{2} + 2\binom{n}{3}.$$

Hence

$$s_n = \frac{n(n-1)}{2} + \frac{2n(n-1)(n-2)}{6} = \frac{n(n-1)(2n-1)}{6}.$$

Since the sum of the squares of the coefficients is $\frac{s_n}{n^2}$ we have

$$\text{Var}[x_t] = \frac{s_n}{n^2}\sigma^2 = \frac{(n-1)}{n}\frac{(2n-1)}{6}\sigma^2.$$

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